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GLOBAL SLIDING MODE OBSERVERS FOR SOME UNCERTAIN NONLINEAR SYSTEMS: DISSIPATIVE APPROACH

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- -

FIRMA

"ADDRESS: The tool implementing the mediation between theory and practice, between thought and observation, is mathematics. Mathematics builds the connecting bridges and is constantly enhancing their capabilities. Therefore it happens that our entire contemporary culture, in so far as it rests on intellectual penetration and utilization of nature, finds its foundations in mathematics.

Already some time ago GALILEO said "Only one who has learned the language and signs in which nature speaks to us can understand nature."

This language however is mathematics, and these signs are the figures of mathematics.

KANT remarked "I maintain that, in any particular natural science, genuine scientific content can be found only in so far as mathematics is contained therein."

In fact we do not have command of a scientific theory until we have peeled away and fully revealed the mathematical kernel. Without mathematics, modern astronomy and physics would be impossible. The theoretical parts of these sciences almost dissolve into branches of mathematics. Mathematics owes its prestige, to the extent that it has any among the general public, to these sciences along with their numerous broader applications. Although all mathematicians have denied it, the applications serve as the measure of worth of mathematics.

GAUSS speaks of the magical attraction which made number theory the favorite science of the first mathematician—not to mention the inexhaustible richness of number theory which far surpasses that of any other field of mathematics.

KRONEKER compares number theorists with the lotus eaters, who, once they started eating this food, could not let go of it.

The great mathematician POINCARE once sharply disagreed with Tolstoy's declaration that the proposition "science for the sake of science" would be silly.

The achievements of industry for example would not have seen the light of the world if only applied people had existed and if uninterested fools had failed to promote these achievements.

The honor of the human spirit, so said the famous Konigsburg mathematician JACOBI, is the only goal of all science. We ought not believe those who today, with a philosophical air and reflective tone, prophesy the decline of culture, and are pleased with themselves in their own ignorance. For us there is no ignorance, especially not, in my opinion, for the natural sciences.

Instead of this silly ignorance, on the contrary let our fate be: "We must know, we will know". "

David Hilbert

Dedicado con mucho cariño
a mis padres Nicolas y Teodora,
y a mi hermano Freddy.

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List of Abbreviations

SM	**S**liding **M**ode
SMO	**S**liding **M**ode **O**bservers
UIO	**U**nknown **I**nput **O**bservers
HOSM	**H**igher **O**rder **S**liding **M**ode
SDS	**S**caled **D**issipative **S**tabilizer
STA	**S**uper **T**wisting **A**lgorithm
GSTA	**G**eneralized **S**uper **T**wisting **A**lgorithm
UP	**U**ncertainty/**P**ertubation
BIBS	**B**ounded-**I**nput-**B**ounded-**S**tate
DOF	**D**egrees **O**f **F**reedom
LTI	**L**inear **T**ime **I**nvariant
w.r.t.	**w**ith **r**espect **t**o

Chapter 1

Introduction

In the automatic control area, a plant is defined as a mechanism or process that needs to be controlled or supervised. For the modeling of the plant, the concept of state variables is frequently used, which represents the more important variables that better describe the system behavior. The knowledge of state variables at each instant of time (by means of their measurement) leads to a better supervision of the plant or a better performance of the controlled system. However, the measurement of all state variables is not always feasible, either due to the nonexistence of an appropriate measuring instrument, or because of its high cost, or due to other factors.

To deal with the problem of not being able to directly know some or all the state variables, a state estimator or observer is used. A state observer is a dynamic system based on a model of the plant that uses the available information from its inputs and outputs in order to provide estimated states that converge to the real state values of the plant.

The presence of internal or external perturbations, parametric uncertainties, in the following will be referred to simply as uncertainty/perturbation (UP) and the plant model as the uncertain system, gives rise to the problem of designing unknown input observers (UIO). The UIO are estimators or observers of states that provide estimates which convergent to the real state variables of the plant, in spite of the presence of UP considered as the unknown input of the observers.

This work deals with the problem of designing UIO for uncertain nonlinear systems using sliding-mode techniques and a dissipative approach. The sliding-modes and the dissipative approach are some of the techniques most used to design UIO, because of their properties of robustness. The dissipative observers unify well-known observers, e.g. the Luenberger-like observer and the High-gain observer, which provide global exponential convergence to the real states when their required conditions are satisfied. Among these conditions, there is the relative-degree-one restriction, which means that the arbitrary UP affect only the first time derivative of a measured output of the plant. The sliding-mode observers (SMO) provide finite-time convergence of the output estimation error and asymptotic convergence to the real states. These robustness properties of SMO are obtained when the UP is bounded and the bounded-input-bounded-state (BIBS) property in the plant is satisfied. In the next section, a review of the literature is given.

1.1 Observers construction: art state

1.1.1 Dissipative approach on the observer design

The dissipativity is an energy concept with a direct physical interpretation: a system is dissipative if the energy it stores is less than that which is supplied to it (Byrnes, Isidori, and Willems, 1991). In the framework of dynamical systems, the energy or power supplied to a system is proportional to the product of the input and the output, in this sense dissipativity, which is a generalization of passivity, is an input-output concept. The dissipativity plays a fundamental role in control theory (Brogliato et al., 2007), and the idea of using dissipativity concepts for the design of observers has been used in Moreno, 2001; Shim, Seo, and Teel, 2003; Moreno, 2004; Moreno, 2005. Moreno, 2004 uses the dissipativity theory to counteract the nonlinearity effects in the basic linear observer design, where the design methodology generalizes and unifies several standard design methods such as Luenberger-like observer (Ciccarella, Mora, and Germani, 1993; Rajamani, 1998); the use of linear matrix inequalities to design the observer gains for globally Lipschitz (Ciccarella, Mora, and Germani, 1993; Rajamani, 1998; Zemouche and Boutayeb, 2013), or monotonic nonlinear systems (Arcak and Kokotovic, 2001); High-gain observer (Atassi and Khalil, 1999). Moreover, the class of dissipative systems can be expanded, even for systems that are not dissipative by nature, but that under certain conditions can become dissipative, either through state feedback or output (Byrnes, Isidori, and Willems, 1991; Rocha-Cózatl and Moreno, 2001).

In the case of Linear Time Invariant (LTI) systems with UP, the existence problem of UIO, when the UP is arbitrary, has been studied for a long time, where strong* detectability (strong detectability and relative degree one) is a necessary and sufficient condition to ensure the existence of an observer (Hautus, 1983). Consequently, many different design methodologies are known (Hou and Muller, 1994; Chu, 2000; Saberi, Vogel, and Sannuti, 2000). The existence problem of OUI for LTI system with arbitrary UP has been also analyzed under a dissipativity approach, by Moreno, 2001, where the existence of an observer is equivalent to the possibility of rendering the plant dissipative by output injection.

In nonlinear systems with UP, the conditions for the existence of UIO are not very well established as is the case for LTI systems (Hautus, 1983; Hou and Muller, 1994; Moreno, 2001). For a class of nonlinear systems with UP, Seliger and Frank, 1991; Moreno, 2000 propose a method to construct an observer and some conditions for its existence are given. A similar analysis to the case of liner systems (Moreno, 2001) derives an incremental dissipativity property for the multiple-input-multiple-output nonlinear systems with arbitrary UP, where the possibility of rendering the plant dissipative by output injection, which includes the satisfaction of relative-degree-one condition, is a sufficient condition for the existence of OUI (Rocha-Cózatl and Moreno, 2004). This existence condition can be made computable for the observer design, where the required storage function and the output injection can be calculated, under generic conditions, by Linear Matrix Inequalities (Rocha-Cózatl, Moreno, and Zeitz, 2005; Moreno, 2008a; Angulo, Moreno, and Lazaro, 2010; Rocha-Cózatl and Moreno, 2011). The dissipative approach is also applicable to systems with discontinuous or multivalued nonlinearities (Osorio and Moreno, 2006; Guzman-Baltazar and Moreno, 2010). A dynamical interpretation of strong observability and detectability concepts for nonlinear systems with UP is made in Moreno, Rocha-Cózatl, and Wouwer, 2014, which allows dealing with the observer existence problem.

The dissipative approach on the observer design has applications on areas as chemical reactor, biochemical processes and tubular reactors (Schaum et al., 2008; Moreno, 2008b; Schaum, Moreno, and Alvarez, 2008a; Schaum, Moreno, and Alvarez, 2008b; Moreno, Rocha-Cózatl, and Wouwer, 2014).

The relative-degree-one condition is necessary for the existence of UIO in linear and nonlinear systems with arbitrary UP (Hautus, 1983; Moreno, 2001; Moreno, 2004). But in the realistic framework the presence of UP with relative degree grater than one is given in many systems, e.g. mechanical systems (Davila, Fridman, and Levant, 2005; Shtessel, Shkolnikov, and Levant, 2007; Nehaoua et al., 2014), electromechanical systems (Utkin, Guldner, and Shi, 2009). When the relative degree is grater than one it is necessary to know some characteristic of the UP, e.g. the UP is bounded, for designing observers.

If the UP is bounded, the dissipative observers may provide the convergence of the estimation error to a bounded region around the origin (Moreno, 2005) and a possible amplification of measurement noise (Ahrens and Khalil, 2009). But for this case, a theoretical exact convergence of the estimation error output, the estimation error precision under measurement noise, and a reconstruction of the bounded UP are not obtained.

1.1.2 Sliding-mode differentiators/observers

One of the techniques most commonly used for designing UIO, when the UP is bounded and with a relative degree greater than one, is the sliding modes. The sliding-mode observers (Barbot, Djemai, and Boukhobza, 2002; Edwards, Spurgeon, and Tan, 2002; Yan and Edwards, 2007; Shtessel et al., 2014) have proven to be efficient for providing theoretically exact finite-time convergence of the output estimation error, asymptotic convergence to the real states, and in some cases even the reconstruction of the UP.

In LTI systems with bounded UP and under the strong detectability/observability condition the design of sliding-mode observers (SMO) has been extensively studied in the last decade (see e.g. Bejarano and Fridman, 2010; Bejarano, Pisano, and Usai, 2011; Fridman, Levant, and Davila, 2007; Fridman, Davila, and Levant, 2011) to overcome the relative-degree-one restriction with respect to (w.r.t.) the UP (Hautus, 1983; Moreno, 2001). However, one drawback of the SMO is that, most of them, need the state vector affected by UP to be uniformly bounded, here the bounded-input-bounded-state (BIBS) property is required. To overcome the BIBS restriction for LTI systems with bounded UP, the work in Fridman, Levant, and Davila, 2007 proposes a strategy where the Luenberger observer driving the estimation error to a bounded region of the origin, in cascade with a high-order sliding mode (HOSM) differentiator allows global theoretically exact finite-time estimation of the system states. The applicability of this strategy is not clear for the case of nonlinear systems.

In nonlinear systems with bounded UP and with relative degree higher or equal to one, the sliding-mode observers provide theoretically exact convergence to the real system states (Floquet and Barbot, 2007; Fridman et al., 2008; Barbot and Floquet, 2010; Ríos et al., 2015) where the BIBS property is required. The BIBS property allows dealing with nonlinear terms, which are not necessarily Lipschitz, e.g. the quadratic nonlinearity terms obtained from Coriolis and centrifugal forces in the mechanical systems (Xian et al., 2004; Davila, Fridman, and Levant, 2005; Rosas, Alvarez, and Fridman, 2007).

The crucial point for the success of SMO is that they bring with them an implicit or explicit use of a differentiation process, which is the key to why the relative-degree-one restriction can be surpassed. One of the most well-known SM differentiators, which is also the first SM differentiator to appear in the literature, is the Super-Twisting Algorithm (STA) based SM differentiator (Levant, 1998). This SM differentiator is defined as

$$\begin{cases} \dot{z}_1 = -1.5L^{1/2}|z_1 - f(t)|^{1/2}\, sign(z_1 - f(t)) + z_2, \\ \dot{z}_2 = -1.1L\, sign(z_1 - f(t)), \end{cases} \tag{1.1}$$

which guarantees a finite-time estimation of time derivative of $f(t)$ when $|\ddot{f}(t)| \leq L$, where $z_1 - f(t)$ and $z_2 - \dot{f}(t)$ are robustly driven to zero in finite-time. In the presence of deterministic noise bounded by a constant $\varsigma > 0$, the precision for estimation of time derivative is $O(\sqrt{\varsigma})$. Generalizations of the STA based differentiator (Levant, 1998) were presented in Moreno, 2009; Moreno, 2011; Cruz-Zavala, Moreno, and Fridman, 2011, and from these works some SM differentiators of arbitrary order are obtained (Levant, 2003; Angulo, Moreno, and Fridman, 2013).

The SMO have applications in specific problems, e.g. robust and early detection of oscillatory failure case for the electrical flight control system of new generation aircraft (Efimov et al., 2012); estimate Lean and Steering motorcycle dynamics (Nehaoua et al., 2014); missile guidance application (Shtessel, Shkolnikov, and Levant, 2007); wind energy conversion optimization (Evangelista et al., 2013), and analysis and identification of vehicle dynamics (Imine et al., 2011).

Conclusion. The dissipative observers and sliding mode observers require restrictive conditions for their good performance in presence of UP: the BIBS condition for SM observers and relative degree one for dissipative observers, see Table 1.1. The following motivation example illustrates the importance of these conditions.

Observers	UP	Relative degree	BIBS property	Estimation	Type of observer
Dissipative	arbitrary	one	no required	asymptotic	global
Sliding mode	bounded	greater than one	required	in finite time	local

TABLE 1.1: Conditions and properties of the sliding-mode observers and dissipative observers.

1.1.3 Motivational example

Consider the system (1.2), which illustrates the loss of the effectiveness of sliding-mode differentiators and dissipative observers,

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \frac{\sin^2(0.5x_1)}{2(1+\cos^2(x_1))}x_2 - 9.8\frac{\sin(0.2x_1)}{4(1+\cos^2(x_1))} + w(t), \end{cases} \tag{1.2}$$

where $x_1, x_2 \in \mathbb{R}$ are the states, x_1 is the measurable state, $w \in \mathbb{R}$ is an UP. The system (1.2) does not have BIBS property. The non linearity $\varphi(x_1, x_2) = \frac{\sin^2(0.5x_1)}{2(1+\cos^2(x_1))}x_2 - 9.8\frac{\sin(0.2x_1)}{4(1+\cos^2(x_1))}$ is not globally Lipschitz (because $\frac{\partial\varphi}{\partial x_1}$ is not uniformly bounded), but it is Lipschitz with respect to the second variable with Lipschitz constant $L_\varphi = 1/2$. This system has relative degree two w.r.t. the measured output x_1 and the UP as $w(t)$.

Sliding-mode observer. Applying the Levant's differentiator (Levant, 1998) in (1.1) to the measured variable x_1 of the system (1.2), one obtains the following expression

$$\begin{cases} \dot{z}_1 = -1.5L^{1/2}|z_1 - x_1|^{1/2} sign(z_1 - x_1) + z_2, \\ \dot{z}_2 = -1.1L sign(z_1 - x_1). \end{cases} \tag{1.3}$$

The simulations with $L = 8$ for the gains of (1.3), initial conditions $(x_1(0), x_2(0)) = (-0.5, 6.3)$, $(z_1(0), z_2(0)) = (3, 3)$ and the UP as

$$w(t) = \sin(2t) + 0.5\sin(3t)cos(t) - 1.5cos(t) + 1, \tag{1.4}$$

show that the estimation state z_2 converges to the true state x_2 in finite time at $t = 1.47$. But this finite-time convergence is affected from $t = 3.3$, see Figure 1.1.

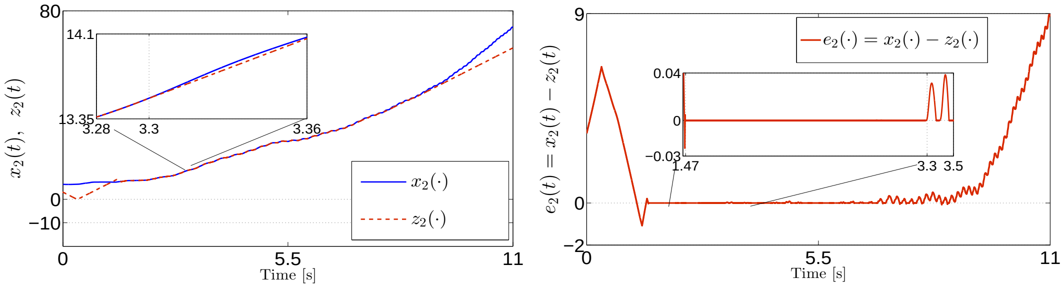

FIGURE 1.1: The estimation state $z_2(t)$ in the sliding-mode differentiator (1.3), and the true state $x_2(t)$ in (1.2) and its estimation error $e_2 = x_2 - z_2$.

This happens because the magnitude of the nonlinearity along the trajectories $\varphi(x_1(t), x_2(t))$ together with $w(t)$ exceeds the gain of 8.8 of the discontinuous term in the differentiator (1.3), see Figure 1.2.

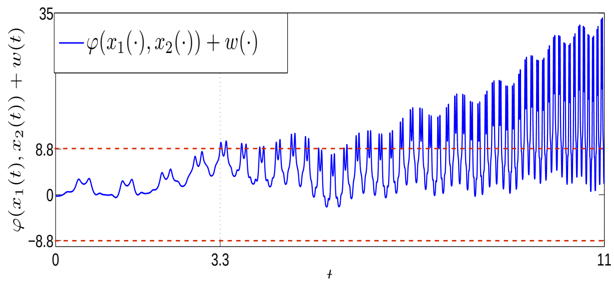

FIGURE 1.2: The function $\varphi(x_1(t), x_2(t)) + w(t)$ exceeds the gain 8.8 of the discontinuous term in the differentiator (1.3)

Dissipative observer. The nonlinearity $\varphi(x_1, x_2)$ in (1.2) satisfies a dissipative property, since the nonlinearity $\tilde{\varphi}(x_1, x_2, h) := \varphi(x_1, x_2) - \varphi(x_1, x_2 + h)$ defined from $\tilde{\varphi}$ is $\{-1, 0, L_\varphi^2\} - dissipative$, i.e.

$$\begin{bmatrix} \tilde{\varphi} \\ h \end{bmatrix}^T \begin{bmatrix} -1 & 0 \\ 0 & L_\varphi^2 \end{bmatrix} \begin{bmatrix} \tilde{\varphi} \\ h \end{bmatrix} \geq 0 \text{ for all } h \in \mathbb{R}, \text{ is satisfied.}$$

Applying the observer in the dissipative approach (Rocha-Cózatl and Moreno, 2011) one obtains:

$$\begin{cases} \dot{v}_1 = v_2 + l_1(v_1 - y), \\ \dot{v}_2 = \varphi(x_1, v_2 + l_3(v_1 - y)) + l_2(v_1 - y), \end{cases} \tag{1.5}$$

where the parameters $l_1 = -1.74$, $l_2 = -3.57$, $l_3 = -0.38$ are obtained as it is proposed in Rocha-Cózatl and Moreno, 2011. The simulations with initial conditions $(x_1(0), x_2(0)) = (-0.5, 6.3)$, $(v_1(0), v_2(0)) = (30, 30)$ and the UP as in (1.4), show that the estimated trajectory v_2 in (1.5) does not converge to the true state x_2 of (1.2); but it converges to some neighborhood of x_2, see Figure 1.3.

This motivational example illustrates the lack of effectiveness of both the sliding-mode and dissipativity approaches for the observer design when the system does

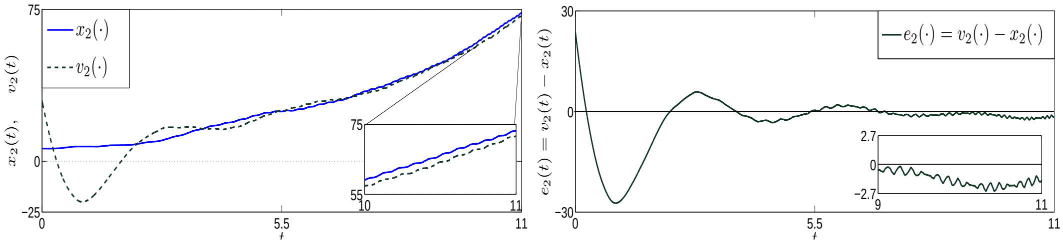

FIGURE 1.3: The estimation state $v_2(t)$ in (1.5) and the true state $x_2(t)$ in (1.2) and its observation error $e_2 = v_2 - x_2$

not have the BIBS property and has a relative degree greater than one w.r.t. the bounded UP.

1.2 Problem statement

The problem of the design of observers for nonlinear systems in presence of uncertainty and perturbation is addressed, where the UP is bounded, the BIBS property is not required, and the relative degree is greater than one. In this sense, the following classes of nonlinear systems are considered.

1.2.1 Chain of integrators with uncertainty and perturbation

Consider the following chain of integrators with nonlinear terms and UP,

$$\Sigma_1 : \begin{cases} \dot{x} &= Ax + B\,(\psi(x) + w(t,x)) + \varphi(u,y), \\ y &= Cx, \end{cases} \tag{1.6}$$

where

$$A := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad B := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{n\times 1}, \quad C := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}^T \in \mathbb{R}^{1\times n}.$$

The state vector is $x \in \mathbb{R}^n$; $y \in \mathbb{R}$ is the measured output; $\psi(\cdot,\cdot) \in \mathbb{R}$ and $\varphi(\cdot,\cdot) \in \mathbb{R}^n$ are continuous nonlinearities; $u \in \mathbb{R}^p$ is the control input; $w(t,x) \in \mathbb{R}$ is the bounded UP, i.e. $|w(\cdot)| \leq \varrho_w$. Assume that the system (1.6) is forward complete, i.e. the trajectories do not escape to infinity in finite time, and the system (1.6) is not required to be BIBS.

The chain of integrators is a very versatile and well studied form since all linear controllable systems and many nonlinear ones can be transformed to it through a coordinate transformation. Some examples can be found: nonlinear observers with approximately linear error dynamics (Banaszuk and Sluis, 1997; Nicosia, Tomei, and Tornambe, 1988; Lynch and Bortoff, 2001); uniformly (in the input) observable nonlinear systems (Gauthier and Bornard, 1981; Gauthier, Hammouri, and Othman, 1992) and into this particular form (Khalil, 2002; Polyakov, Efimov, and Perruquetti, 2016).

The works in Gauthier, Hammouri, and Othman, 1992; Lynch and Bortoff, 2001; Moreno, 2004; Moreno, 2005; Rocha-Cózatl and Moreno, 2011 provide observers with asymptotic convergence to unmeasurable states of (1.6) for the case without UP, otherwise, they are not applicable. The results in Barbot, Boukhobza, and Djemai, 2003; Levant, 2003; Angulo, Moreno, and Fridman, 2013 provide observers of theoretically exact convergence in finite time to unmeasurable states of (1.6) under condition BIBS for bounded UP.

The aim of this work is to provide a global exact finite-time convergent observer for the system (1.6), where the UP is bounded and the system may not have the BIBS property. For this purpose, the proposed observer combines the components of a dissipative observer (Rocha-Cózatl and Moreno, 2011) and a HOSM differentiator (Levant, 2003) under a cascade scheme taking advantage of the dissipative properties (Moreno, 2004) that could provide the nonlinearities in the system.

When $n = 2$ in (1.6), the resulting class of system coincides with the nonlinear mechanical systems with one degree of freedom (1-DOF) where the nonlinearities are obtained from Coriolis force and frictions. This class of uncertain nonlinear system will be analyzed.

1.2.2 1-DOF mechanical systems with uncertainty and perturbation

Consider the following second-order system

$$\Sigma : \begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \psi_m(x_1, x_2) + \varphi_m(u, y) + w(t, x), \end{cases} \tag{1.7}$$

which is obtained from the following 1-DOF mechanical system with UP

$$m(q)\ddot{q} + c(q)\dot{q}^2 + g(q) + \psi(q, \dot{q}) + h \cdot \dot{q} + \lambda \cdot \text{sign}(\dot{q}) = u + \tilde{\delta}(t, q, \dot{q}), \tag{1.8}$$

where $x_1 = q$, $x_2 = \dot{q}$ and

$$\begin{aligned} \psi_m(x_1, x_2) &= -m^{-1}(x_1)c(x_1)x_2^2 - m^{-1}(x_1)\psi(x_1, x_2) - m^{-1}(x_1)h \cdot x_2, \\ \varphi(u, y) &= -m^{-1}(y)g(y) + m^{-1}(y)u, \\ w(t, x_1, x_2) &= -m^{-1}(x_1)\lambda sign(x_2) + m^{-1}(x_1)\tilde{\delta}(t, x_1, x_2). \end{aligned}$$

The state $q \in \mathbb{R}$ is the measured position; $m(q) \in \mathbb{R}$ is the inertia term; $c(q)\dot{q}^2$ represents Coriolis and centrifugal forces; ψ is a continuous nonlinearity (e.g. other types of frictions, air resistance, etc.); the parameters $h, \lambda \in \mathbb{R}$; $h \cdot \dot{q}$ and $\lambda \cdot sign(\dot{q})$ are viscous and dry frictions; $g(q)$ denotes gravitational forces; $\tilde{\delta}(t, q, \dot{q})$ is an UP and $u \in \mathbb{R}$ is the control input.

The following classes of mechanical systems will be considered.

1.2.3 2-DOF mechanical systems with uncertainty and perturbation

Consider the nonlinear system

$$\Sigma_2 : \begin{cases} \dot{x} = & z, \\ \dot{z} = & \psi_M(x, z) + \varphi(u, y) + w(t, x, z), \\ y = & x, \end{cases} \tag{1.9}$$

which is obtained from the following mechanical system with UP

$$M(x)\ddot{x} + C(x, \dot{x})\dot{x} + G(x) + \psi(x, \dot{x}) + H\dot{x} + \Lambda sign(\dot{x}) = Du + \tilde{\delta}(t, x, \dot{x}), \tag{1.10}$$

where

$$\psi_M(x,z) = -M^{-1}(x)C(x,z)z - M^{-1}(x)\psi(x,z) - M^{-1}(x)Hz,$$
$$\varphi(u,y) = -M^{-1}(y)G(y) + M^{-1}(y)Du,$$
$$w(t,x,z) = -M^{-1}(x)\Lambda sign(z) + M^{-1}(x)\tilde{\delta}(t,x,z).$$

The state $x \in \mathbb{R}^2$ is the measured position; $M(x) \in \mathbb{R}^{2\times 2}$ is the inertia matrix; $C(x,\dot{x})\dot{x}$ represents Coriolis and centrifugal forces; ψ is a continuous nonlinearity (e.g. other types of frictions, air resistance, etc.); the matrices $H, \Lambda, D \in \mathbb{R}^{2\times 2}$; $H\dot{x}$ and $\Lambda sign(\dot{x})$ are viscous and dry frictions; $G(x)$ denotes gravitational forces; $\tilde{\delta}(t,x,\dot{x})$ is an UP and $u \in \mathbb{R}^2$ is the control input.

One of the problems on the global observer design for (1.8), (1.10) has been dealing with the nonlinearities of Coriolis and centrifugal forces, which have attracted the attention of several researchers (Besançon, 2000; Mabrouk, Mazenc, and Vivalda, 2004; Astolfi, Ortega, and Venkatraman, 2010; Stamnes, Aamo, and Kaasa, 2011) for the case without frictions and UP. Through state transformations, Astolfi, Ortega, and Venkatraman, 2010; Stamnes, Aamo, and Kaasa, 2011 propose observers with fairly high dimension, namely $3n+1$ and $2n+2$ respectively, where n is the dimension of the unmeasured velocity. Besançon, 2000; Mabrouk, Mazenc, and Vivalda, 2004 propose observers with the same dimension of the system for a class of mechanical systems.

The challenge of dealing with viscous and dry frictions, UP can be tried by SM observers/differentiators (Levant, 1998; Xian et al., 2004; Davila, Fridman, and Levant, 2005; Moreno, 2009) providing local observer with convergence theoretically exact in finite time to the velocity, where BIBS property in the system is required.

The aim is to provide a global exact finite-time convergent observer for the systems (1.8), (1.10), where the UP is bounded and the system may not have the BIBS property. For this aim, the proposed observer combines the components of a dissipative observer (Rocha-Cózatl and Moreno, 2011) and a HOSM differentiator (Levant, 2003) using the Generalized Super Twisting Algorithm (GSTA) (Moreno, 2009) taking advantage of the dissipative properties (Moreno, 2004) that could have the nonlinearities in the system.

1.3 Main contributions of this work

Two classes of nonlinear systems (a chain of integrators and a class of mechanical systems) were considered, with uncertainties and perturbations, which have been extensively studied in the literature for the case without UP. This work consider a bounded UP which does not necessarily vanish. The difference between this work and other works that use the sliding-mode technique (Xian et al., 2004; Davila, Fridman, and Levant, 2005; Floquet and Barbot, 2007; Fridman et al., 2008; Polyakov, Efimov, and Perruquetti, 2016) is that here global observers conserving the finite-time convergence to the unmeasured states are provided. This work considers nonlinear systems with relative degrees higher than one, where the use of dissipative properties of nonlinearities for the observer design is maintained as it is used by the dissipative observer for nonlinear system with relative degree one (Rocha-Cózatl and Moreno, 2011), see Table 1.2.

Observers	UP	Relative degree	BIBS property	Estimation	Type of observer
Dissipative	arbitrary	one	no required	asymptotic	global
Sliding mode	bounded	greater than one	required	in finite time	local
Proposed	bounded	greater than one	no required	in finite time	global

TABLE 1.2: Conditions and properties of the sliding-mode observers, dissipative observers and proposed observers.

This work proposes two structures of observers:

i) The first one uses the terms of the dissipative observer and a HOSM differentiator under a cascade scheme. It permits to show that the dissipative observer and a HOSM differentiator can work together under a cascade scheme such as in the linear case (Fridman, Levant, and Davila, 2007) with the Luenberger observer and a HOSM differentiator. This structure lets us use the Lyapunov functions of each one providing convergence in finite time to the real states. This structure allows considering nonlinear systems with relative degrees higher than one w.r.t. the UP. A modification of the dissipative observer is given for a chain of integrators with nonlinearities and UP, and is called scaled dissipative stabilizer. This one allows to obtain uniformly ultimately bounded where it is shown that increasing the gain of the observer it is possible to reduce the ultimate bound. This allows to finally improve the independence of the gains between the scaled dissipative stabilizer and a HOSM differentiator in a cascade scheme.

ii) The second one introduces correction terms in the nonlinearities and it uses the Generalized Super-Twinting algorithm (Moreno, 2009) which combines the terms of the dissipative observer and the Super-Twinting algorithm. Unlike the first structure in the case of second-order systems, the second structure allows more flexibility in the design of gains. A Lyapunov function which considers the dissipative properties of the nonlinearities in the system and ensures finite-time convergence is proposed. Unlike global observers for mechanical systems (Besançon, 2000; Mabrouk, Mazenc, and Vivalda, 2004; Astolfi, Ortega, and Venkatraman, 2010; Stamnes, Aamo, and Kaasa, 2011) which only consider the problem of Coriolis, this structure is applied to a class of mechanical systems with terms of Coriolis, viscous and dry frictions and UP.

The gain design for the observers is performed through matrix inequalities, for which their feasibility is assured. In the experimental area, an implementation test on the pendulum-cart system (INTECO, 2008) for one of the proposed observers is performed.

Partial results of this work have been reported in international conferences (Apaza-Perez, Fridman, and Moreno, 2015; Apaza-Perez, Moreno, and Fridman, 2016) and scientific journals (Apaza-Perez, Fridman, and Moreno, 2017; Apaza-Perez, Moreno, and Fridman, 2018).

This work is organized as follows. A chain of integrators is considered in Chapter 2, where a scaled dissipative stabilizer is proposed which combined with a HOSM differentiator under a cascade scheme provides a global observer. Nonlinear mechanical systems of one-degree-of-freedom and two-degree-of-freedom are considered in Chapter 3 and Chapter 4, respectively. In these classes of nonlinear mechanical systems, Coriolis and centrifugal forces, dry and viscous frictions, perturbations with relative degree two were considered. In Chapter 5, one of the proposed observers is used in the experimental framework through a car-pendulum system where its effectiveness, robustness, and applicability are shown.

Chapter 2

HOSM observer with a scaled dissipative stabiliser for a chain of integrators of arbitrary order

2.1 Introduction

The chain of integrators is a very versatile and well studied system. All linear controllable systems and many nonlinear ones can be transformed into this form through a coordinate transformation. Some examples of these transformations are nonlinear observers with approximately linear error dynamics (Banaszuk and Sluis, 1997; Nicosia, Tomei, and Tornambe, 1988; Lynch and Bortoff, 2001); uniformly (in the input) observable nonlinear systems (Gauthier and Bornard, 1981; Gauthier, Hammouri, and Othman, 1992) and into this particular form (Khalil, 2002; Polyakov, Efimov, and Perruquetti, 2016).

The works in Gauthier, Hammouri, and Othman, 1992; Lynch and Bortoff, 2001; Moreno, 2004; Moreno, 2005; Rocha-Cózatl and Moreno, 2011 provide observers that converge asymptotically to the unmeasurable states of (1.6), only for the case without UP, otherwise, they lose the good performance. The results in Barbot, Boukhobza, and Djemai, 2003; Levant, 2003; Angulo, Moreno, and Fridman, 2013 provide observers with theoretically exact finite-time convergence to unmeasurable states of (1.6), under the BIBS condition with respect to bounded UP.

The aim of this chapter is to present a global exact finite-time convergent observer for a chain of integrators of arbitrary order with nonlinear terms and a bounded UP, and which might not have the BIBS property. It is shown and proved that the direct cascade connection of the dissipative observer with the HOSM differentiator provides a theoretically exact estimation in finite time of the real states. This direct connection presents some drawbacks due to the fact that the gains of the HOSM differentiator grow along with the dissipative observer gains. That is why a scaled dissipative stabiliser (SDS) is proposed. This SDS ensures that the HOSM differentiator gains can be chosen, according to the upper-bound of the unknown input. Also, the scaled dissipative stabiliser gains can grow and do not affect the HOSM differentiator gains. Consequently, the global finite-time exact convergence of the cascaded scaled dissipative-HOSM differentiator observer can be achieved.

The rest of this chapter is organized as follows. In Section 2.2 the problem statement is shown. In Section 2.3, the dissipative observer and a cascade connection with the HOSM differentiators are exposed. In Section 2.4, a scaled dissipative stabiliser is introduced. In Section 2.5, the proposed global exact observer and the main result are exposed. The main results are applied to examples in order to show its effectiveness. In Section 2.6, some conclusions are provided. The proofs of all results are in Appendix A.

2.2 Problem Statement

Consider the following chain of integrators with nonlinear terms and UP,

$$\Sigma : \begin{cases} \dot{x} & = A\,x + B\,\left(\psi(x) + w(t,x)\right) + \varphi(u,y), \\ y & = Cx, \end{cases} \tag{2.1}$$

where

$$A := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad B := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{n\times 1}, \quad C := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}^T \in \mathbb{R}^{1\times n}.$$

The state vector $x \in \mathbb{R}^n$; $y \in \mathbb{R}$ is the measured output; $\psi(\cdot,\cdot)$ and $\varphi(\cdot,\cdot)$ are continuous nonlinear functions; $u \in \mathbb{R}^p$ is the control input; $w(t,x) \in \mathbb{R}$ is a bounded UP, e.i. $|w(\cdot)| \leq \varrho_w$. Assume that system (2.1) is forward complete, i.e. the trajectories do not escape to infinity in finite time. Note that system (2.1) is not required to be BIBS.

The aim of this paper is to present a global exact finite-time convergent observer for system (2.1). This work proposes a scaled dissipative stabiliser ensuring that, when the HOSM differentiator gains are chosen according to the upper-bound of the UP, the SDS gains can grow without affecting the HOSM differentiator gains and global finite-time exact convergence of the cascaded scaled dissipative-HOSM differentiator can be achieved. The idea of the proposed observer is illustrated in Figure 2.1.

2.3 HOMS and dissipative observers under a cascade scheme

The dissipative observer was introduced in Moreno, 2004, Rocha-Cózatl and Moreno, 2011 the following form

$$\dot{v} = Av + B\psi(v + N(Cv - y)) + K(Cv - y) + \varphi(u,y). \tag{2.2}$$

It considers the structure of system (2.1) along with two terms containing the output error injection $(Cv - y)$. The first one appears additively in the system and the second one appears in the argument of the nonlinear term ψ.

Definition 1 (Rocha-Cózatl and Moreno, 2004)**.** *A nonlinearity time variant* $\gamma : [0,\infty)\times R^p \to R^m$*, piecewise continuous in* t*, locally Lipschitz in* ν *such that* $\gamma(t,0) = 0$ *is called* $\{Q,S,R\}-$dissipative*, if for each* $t \geq 0$ *and* $\nu \in R^p$

$$\begin{bmatrix} \gamma(t,\nu) \\ \nu \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \gamma(t,\nu) \\ \nu \end{bmatrix} \geq 0, \tag{2.3}$$

where $Q \in \mathbb{R}^{m\times m}$*,* $S \in \mathbb{R}^{m\times p}$*,* $R \in \mathbb{R}^{p\times p}$*, and* Q*,* R *are symmetric matrices.* ◇

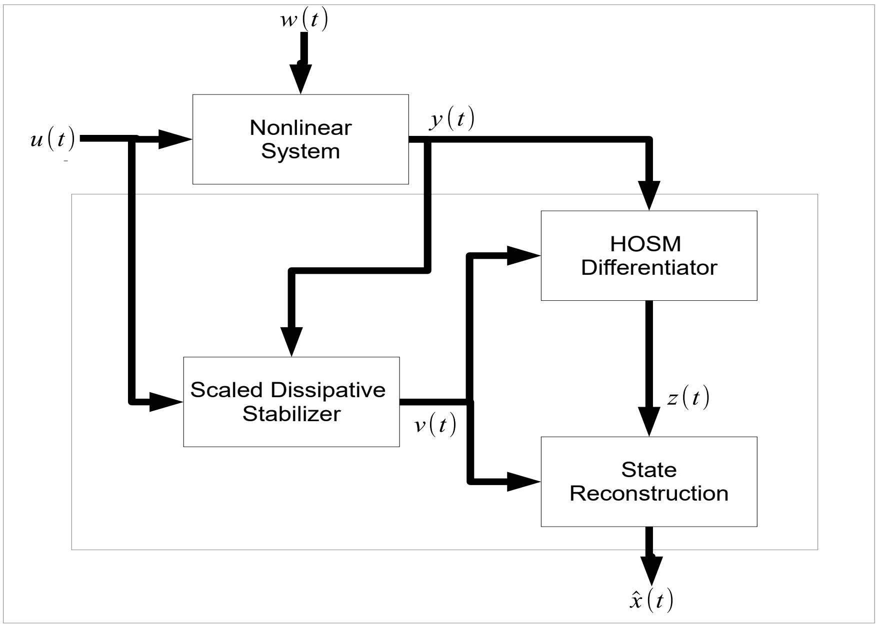

FIGURE 2.1: Proposed global exact observer

Assumption 1. *There exists $q < 0$, $S \in \mathbb{R}^{1\times(n-1)}$ and $R \in \mathbb{R}^{(n-1)\times(n-1)}$ such that the nonlinearity*

$$\Psi(x,h) := \psi(x_1, x_2 + h_1, ..., x_n + h_{n-1}) - \psi(x_1, x_2, ..., x_n), \tag{2.4}$$

is $\{q, S, R\}-$dissipative, i.e.

$$\begin{bmatrix}\Psi(x,h)\\ h\end{bmatrix}^T \begin{bmatrix} q & S\\ S^T & R\end{bmatrix}\begin{bmatrix}\Psi(x,h)\\ h\end{bmatrix} \geq 0, \ \ \forall\, x \in \mathbb{R}^n, \ \ h \in \mathbb{R}^{n-1}. \tag{2.5}$$

◇

Lemma 1. *If Assumption 1 is satisfied, then $\Psi(x,h)$ is $\{-q^2, 0, \tilde{R}\}-$dissipative with*

$$\tilde{R} = \left(\{-\lambda_{\max}(R)q + \lambda_{\max}(S^TS)\}^{1/2} + \{\lambda_{\max}(S^TS)\}^{1/2}\right)^2 I_{n-1}, \tag{2.6}$$

i.e.

$$\begin{bmatrix}\Psi\\ h\end{bmatrix}^T \begin{bmatrix} -q^2 & 0\\ 0 & \tilde{R}\end{bmatrix}\begin{bmatrix}\Psi\\ h\end{bmatrix} \geq 0.$$

▲

The proof of Lemma 1 is in Appendix A.1.

Lemma 1 shows that Assumption 1 implies that the nonlinearity $\psi(\cdot)$ in (2.1) is uniformly Lipschitz with respect to the last $(n-1)$-variables, i.e. there exists a constant $\varrho_\psi > 0$ such that

$$|\psi(x_1, x_2{+}h_1, ..., x_n{+}h_{n-1}) - \psi(x_1, x_2, ..., x_n)| \leq \varrho_\psi \|(h_1, ..., h_{n-1})\| \ \ \forall\, x \in \mathbb{R}^n, h \in \mathbb{R}^{n-1}, \tag{2.7}$$

in this case, $\varrho_\psi = \frac{1}{|q|}\left(\{-\lambda_{\max}(R)q + \lambda_{\max}(S^TS)\}^{1/2} + \{\lambda_{\max}(S^TS)\}^{1/2}\right)$.

2.3.1 A direct connection of HOSM and dissipative observers for a chain of integrators of second order

Consider system (2.1) with $n = 2$. The observer obtained by a cascade scheme (see Figure 2.1) formed by the dissipative observer (2.2) with $n = 2$, and a STA based differentiator (1.1) is given by

$$\begin{cases} \dot{v}_1 = v_2 + l_1(v_1 - y), \\ \dot{v}_2 = \psi(x_1, v_2 + l_3(v_1 - y)) + l_2(v_1 - y), \\ \dot{z}_1 = -1.5L_f^{\frac{1}{2}}|z_1 - (v_1 - y)|^{\frac{1}{2}} sign\,(z_1 - (v_1 - y)) + z_2, \\ \dot{z}_2 = -1.1L_f\, sign\,(z_1 - (v_1 - y))\,, \\ \tilde{x}_1 = v_1 - z_1, \\ \tilde{x}_2 = v_2 + l_1(v_1 - y) - z_2, \end{cases} \tag{2.8}$$

where $\tilde{x}_1$, $\tilde{x}_2$ are the estimates of the states x_1 and x_2, respectively.

Theorem 1. *If the nonlinearity ψ, in (2.1) with $n = 2$, is uniformly Lipschitz ($\varrho_\psi > 0$) with respect to the second variable, and there exists a matrix $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} > 0$ and constants ϵ, k_1, k_2, N_1 with $\epsilon > 0$ such that the inequality*[1]

$$\begin{bmatrix} 2(l_1p_1 + l_2p_2) + \varrho_\psi^2 l_3^2 + \epsilon & \star & \star \\ p_1 + l_1p_2 + l_2p_3 + \varrho_\psi^2 l_3 & 2p_2 + \varrho_\psi^2 + \epsilon & \star \\ p_2 & p_3 & -1 \end{bmatrix} \leq 0 \tag{2.9}$$

is satisfied, then

- *the estimation error $e_v = v - x$ of the dissipative observer is uniformly ultimately bounded, with ultimate bound*

$$\|e_v\| < \frac{2\|PB\|\varrho_w}{\epsilon}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}. \tag{2.10}$$

  Also, the second derivative of the measured output of the dissipative error $e_{vy} = Cv - y$ is bounded, with bound

$$L_f = R\left(\varrho_\psi\sqrt{l_3^2 + 1} + \sqrt{(l_1^2 + l_2)^2 + l_1^2}\right) + \varrho_w, \tag{2.11}$$

  where $R = \frac{2\varrho_w\|(p_2,p_3)\|+\delta}{\epsilon}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$ and $\delta > 0$ is an arbitrary constant,

- *the estimated states of the proposed observer (2.8) converge to the true states of system (2.1) in finite time.*

▲

The proof of Theorem 1 is in Appendix A.2.

[1] $(\star)$ indicates that the matrix is a symmetric matrix.

2.3.2 Motivational example of Chapter 1

Recall the example (1.2) given in Section 1.1.3. Assumption 1 is satisfied with $\psi = \varphi$, $q = -1$, $S = 0$, $R = L_{\varphi}^2$. Solving the inequality (2.15), the values found are $p_1 = 1.33$, $p_2 = -0.53$, $p_3 = 0.53$, $l_1 = -1.74$, $l_2 = -3.57$, $l_3 = -0.38$, $\epsilon = 0.5$. Thus $L_f = 31.1149$ with $\delta = 0.0001$. The proposed observer is given by

$$\begin{cases} \dot{v}_1 = v_2 - 1.74(v_1 - y), \\ \dot{v}_2 = \varphi(x_1, v_2 - 0,38(v_1 - y)) - 3.57(v_1 - y), \\ \dot{z}_1 = -1.5(31.1149)^{\frac{1}{2}}|z_1 - (v_1 - y)|^{\frac{1}{2}} sign\,(z_1 - (v_1 - y)) + z_2, \\ \dot{z}_2 = -1.1(31.1149) sign(z_1 - (v_1 - y)), \\ \tilde{x}_1 = v_1 - z_1, \\ \tilde{x}_2 = v_2 - 1.74(v_1 - y) - z_2. \end{cases} \tag{2.12}$$

For the simulations, consider the initial conditions $v_1(0) = 30$, $v_2(0) = 30$, $z_1(0) = 0$, $z_2(0) = 0$ and the UP as in (1.4). The estimated state $\tilde{x}_1$ converges to x_1 in finite time at $t = 2.58$, and the estimated state $\tilde{x}_2$ converges to x_2 at $t = 2.605$ as illustrated in Figure 2.2 and Figure 2.3, respectively.

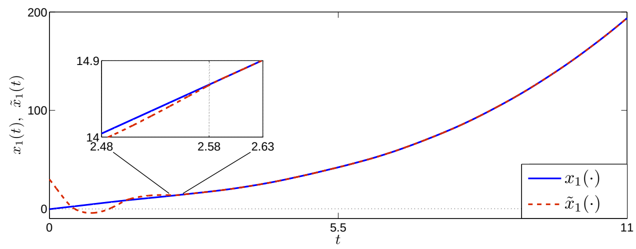

FIGURE 2.2: The estimated state $\tilde{x}_1(t)$ in the observer (2.12) and real state $x_1(t)$ of (1.2)

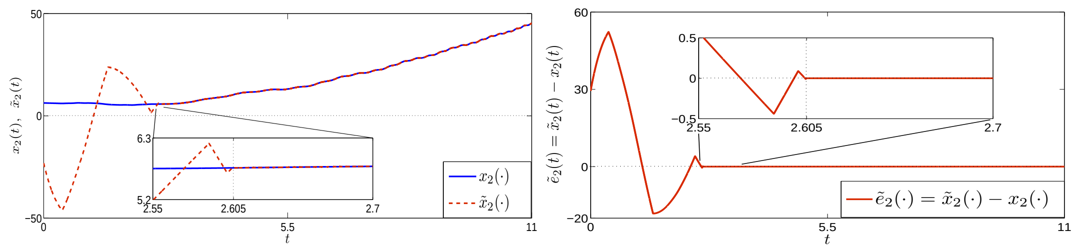

FIGURE 2.3: The estimated state $\tilde{x}_2(t)$ of the observer (2.12) and real state $x_2(t)$ of (1.2), and the estimation error $\tilde{e}_2 = \tilde{x}_2 - x_2$

The observer (2.8) obtained from a cascade structure between the dissipative observer and a HOSM differentiator has the following drawbacks:

i) The existence of a dissipative observer is not ensured because the feasibility of the matrix inequality (2.9) requires a special analysis.

ii) It is not clear if it is necessary to grow the gains l_i to reduce the estimation error of the dissipative observer (2.10).

iii) There is a direct connection between the dissipative observer gains and the estimation error of the highest derivatives defining the HOSM differentiator gain

(2.11): the greater the dissipative observer gains l_i are the greater the differentiator gain L_f should be.

In the next section the scaled dissipative stabiliser will be introduced. This stabiliser has the following two main properties:

- Assumption 1 and the structure of the SDS ensure the design feasibility,
- the gain L_f for the finite-time estimation of the derivatives of measurable output error $e_{vy} = Cv - y$ is not affected when the SDS gains grow.

2.4 Scaled dissipative stabiliser

Let us introduce the structure of the SDS adding a matrix scaling factor, $\Delta_l = diag\{l^i\}$ for $i = 1, ..., n$ to the dissipative observer structure (2.2), in the linear injection of output error:

$$\dot{v} = Av + B\psi(v + N(Cv - y)) + \Delta_l K(Cv - y) + \varphi(u, y), \tag{2.13}$$

where $K,\ N \in \mathbb{R}^n$ with $N = \begin{bmatrix} -1 & N_2 & \cdots & N_n \end{bmatrix}^T$ are design parameters.

Defining $e_v := v - x$ one can obtain the dynamics of the estimation error as

$$\dot{e}_v = (A + \Delta_l KC)e_v + B\left(\psi(v + NCe_v) - \psi(x) - w(t,x)\right). \tag{2.14}$$

Theorem 2. *If Assumption 1 is satisfied, then*

i) there exist matrices $P = P^T > 0$ and $Q = Q^T > 0$ with $P, Q \in \mathbb{R}^{n\times n}$, $K \in \mathbb{R}^n$, $\tilde{N} = \begin{bmatrix} N_2 & \cdots & N_3 \end{bmatrix}^T \in \mathbb{R}^{(n-1)}$ and scalars $\epsilon > 0$, $l_0 \geq 1$ such that the inequality

$$\begin{bmatrix} l\left(PA_K + A_K^T P + \epsilon Q + \frac{1}{l^{2n}}(\tilde{I}_n\Delta_l)_{l\tilde{N}}^T R(\tilde{I}_n\Delta_l)_{l\tilde{N}}\right) & \star \\ B^T P + \frac{1}{l^{2n-1}} S(\tilde{I}_n\Delta_l)_{l\tilde{N}} & lq \end{bmatrix} \leq 0, \tag{2.15}$$

with

$$A_K := A + KC, \quad \text{and} \quad (\tilde{I}_n\Delta_l)_{l\tilde{N}} := \tilde{I}_n\Delta_l + l\tilde{N}C,$$
$$\tilde{I}_n := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{(n-1)\times n},$$

is satisfied for all $l \geq l_0$.

ii) the estimation error (2.14) of the SDS is uniformly ultimately bounded with ultimate bound

$$\|e_v\| < \frac{b}{l}, \quad \text{for all} \ \ l \geq l_0, \tag{2.16}$$

where $b := \frac{2\|PB\|\varrho_w}{\epsilon\lambda_{\min}(Q)}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}n}$. ▲

The scaling factor Δ_l in Theorem 2 ensures the convergence of the estimation error (2.14) to the desired neighborhood of the origin. If the UP is vanishing, Theorem

2 ensures exponential convergence to the origin of the dissipative error dynamics (2.14).

From the proof of Theorem 2 in Appendix A.3, one can derive the following selection rules for the parameters in the matrix inequality (2.15).

Algorithm for the SDS design:

i) Choose K such that A_K is a Hurwitz matrix.

ii) Select $Q = Q^T > 0$, $\tilde{N}$ and $\epsilon > 0$.

iii) Find $P = P^T > 0$ such that the Lyapunov inequality $PA_K + A_K^T P + \epsilon Q + \Lambda_{\varrho_\psi} < 0$, with

$$\Lambda_{\varrho_\psi} := diag\left\{\frac{\varrho_\psi}{(n-i+1)^2}\right\} \quad \text{for } i = 1, ..., n,$$

is satisfied.

iv) Grow the parameter l to fulfill the matrix inequality (2.15).

2.5 Higher-order sliding-mode observers with a scaled dissipative stabiliser

The higher-order sliding-mode observer with scaled dissipative stabiliser proposed in this work consists of two structures: a scaled dissipative stabiliser and a HOSM differentiator (see also Figure 2.1):

$$\dot{v} = Av + B\psi(v + N(Cv - y)) + \varphi(u, y) + \Delta_l K(Cv - y), \tag{2.17a}$$

$$\dot{z} = W(z, Cv - y), \tag{2.17b}$$

$$\hat{x} = v - \mathscr{O}^{-1} z, \tag{2.17c}$$

where $W(\cdot,\cdot)$ corresponds to the HOSM differentiator (Levant, 2003) defined as

$$W(z, Cv - y) := \begin{bmatrix} -\alpha_1 L_f^{1/n} \lceil z_1 - Cv + y \rfloor^{(n-1)/n} + z_2 \\ -\alpha_2 L_f^{1/(n-1)} \lceil z_2 - \dot{z}_1 \rfloor^{(n-2)/(n-1)} + z_3 \\ \vdots \\ -\alpha_{n-1} L_f^{1/2} \lceil z_{n-1} - \dot{z}_{n-2} \rfloor^{1/2} + z_n \\ -\alpha_n L_f \lceil z_n - \dot{z}_{n-1} \rfloor^0 \end{bmatrix}, \tag{2.18}$$

with $\lceil \circ \rfloor^s := |\circ|^s \, sign(\circ)$, and $\mathscr{O}$ defined as

$$\mathscr{O} = \begin{bmatrix} C \\ C(A + \Delta_l KC) \\ \vdots \\ C(A + \Delta_l KC)^{n-1} \end{bmatrix}, \tag{2.19}$$

and Δ_l, K, N, L_f are design parameters and the α_i's are taken as in Levant, 2003.

The gains of the system (2.17a) are designed such that the estimation error dynamics (2.14) converges to a neighborhood of the origin, where its measured output $Cv(t) - y(t)$ has a bounded n-th derivative with bound L_f. This bound L_f ensures the globally exact finite-time convergence of the HOSM (2.17b) to the derivatives of $Cv(t) - y(t)$. Finally, the equation (2.17c) describes the estimated state $\hat{x}$ combining the information of both systems, (2.17a) and (2.17b).

Main result. The following Theorem ensures global exact finite-time of the higher-order sliding-mode observer with a dissipative stabiliser to the real system's states (2.1).

Theorem 3. *Suppose that assumption 1 is satisfied, then*

i) *there exist matrices $P = P^T > 0$ and $Q = Q^T > 0$ with $P, Q \in \mathbb{R}^{n\times n}$, $K \in \mathbb{R}^n$, $\tilde{N} = \begin{bmatrix} N_2 & \cdots & N_3 \end{bmatrix}^T \in \mathbb{R}^{(n-1)}$ and scalars $\epsilon > 0$, $l_0 \geq 1$ such that the matrix inequality (2.15), is satisfied for all $l \geq l_0$,*

ii) *the estimation error (2.14) is uniformly ultimately bounded, where the n-th derivative of the measured output of the estimation error $e_{vy} = Cv - y$ is bounded, with bound*

$$L_f = \left\{ \frac{2\|PB\| + \delta}{\epsilon \lambda_{\min}(Q)} \left[\left\| (A + KC)^n G^T diag \left\{ \frac{\lambda_{\max}(P)}{\lambda_i(P)} \right\} \right\| \sqrt{n} + \right. \right. \tag{2.20}$$

$$\left. \left. + \varrho_\psi \left\| (I_n + \tilde{N}C) \frac{\Delta_l}{l^{n+1}} G^T \right\| \right] + 1 \right\} \varrho_w,$$

where $\varrho_\psi = \frac{1}{-q}\left(\sqrt{-\lambda_{\max}(R)q + \lambda_{\max}(S^T S)} + \{\lambda_{\max}(S^T S)\}^{1/2}\right)$, G is an orthogonal matrix, and $D = diag\{\lambda_i(P)\}$ with $\lambda_i(P)$ the eigenvalues of the matrix P, for $i = 1, ..., n$, such that $P = G^T DG$,

iii) *the estimated state $\hat{x}$ in (2.17c) converges globally exactly and in finite time to the state x of the system (2.1)*

▲

The proof of Theorem 3 is in Appendix A.4 and one can derive the following selection rules of the parameters in the matrix inequality (2.15) and the value of L_f (2.20).

Algorithm for the HOSM observer design with SDS:

i) Choose K such that A_K is a Hurwitz matrix.

ii) Select $Q = Q^T > 0$, $\tilde{N}$ and $\epsilon > 0$.

iii) Find $P = P^T > 0$ such that the Lyapunov inequality $PA_K + A_K^T P + \epsilon Q + \Lambda_{\varrho_\psi} < 0$, with

$$\Lambda_{\varrho_\psi} := diag \left\{ \frac{\varrho_\psi}{(n - i + 1)^2} \right\} \quad \text{for } i = 1, ..., n, \text{ and } \varrho_\psi \text{ defined as in (2.7)},$$

is satisfied.

iv) Grow the parameter l to fulfill the matrix inequality (2.15).

v) Find the orthogonal matrix G and the diagonal matrix D associated with the matrix P such that $P = G^T DG$.

vi) Calculate the parameter L_f in (2.20) and the matrix $\mathscr{O}^{-1}$ in (2.19).

2.5.1 Example

Consider the following third order chain of integrators

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = \psi_1(x_1, x_2, x_3) + w, \end{cases} \tag{2.21a}$$

where the $y := x_1$ is the measurable output and the nonlinearity ψ_1 is

$$\psi_1(x_1, x_2, x_3) := \frac{\sin(0.2x_1)}{2 + 2\cos^2(x_1)} + \frac{\sin^2(x_1) + 1/4}{2 + 2x_2^2} + \frac{\arctan(x_3)}{4} + \frac{5x_3\cos^2(x_1)}{28} + 2. \tag{2.21b}$$

The nonlinearity ψ_1 is greater than 1.5 when the variable x_3 is non negative. If the bound of the UP $w(t)$ is less than one, and the initial condition $x_3(0)$ is non negative, then $\dot{x}_3 > 0.5$ and $x_3(t) \geq 0.5t + x(0)$. Thus, the system (2.21a)-(2.21b) does not have the BIBS property. This is illustrated in Figure 2.4, for initial conditions $x_1(0) = -0.5$, $x_2(0) = 0.5$, $x_3(0) = 1$ and an UP as

$$w(t) := 0.3\sin(2t) - 0.5\cos(\pi t) + 0.2. \tag{2.21c}$$

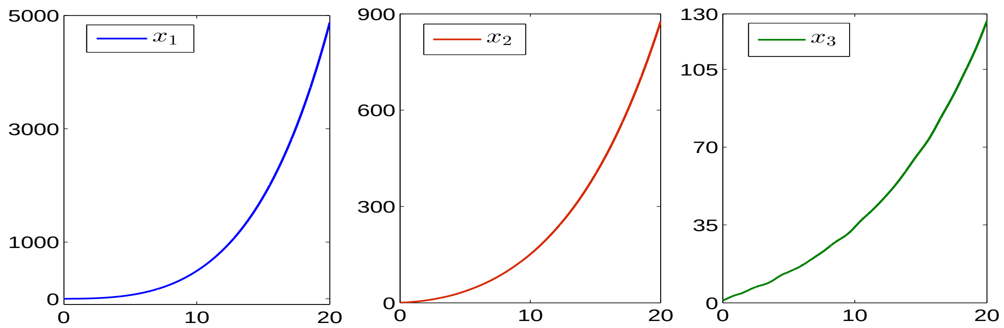

FIGURE 2.4: The states x_1, x_2, x_3 of the system (2.21)

Dissipative observer. The nonlinearity

$$\Psi_1(x_1, x_2, x_3, h_2, h_3) := \psi_1(x_1, x_2 + h_2, x_3 + h_3) - \psi_1(x_1, x_2, x_3), \tag{2.22}$$

defined from the nonlinearity in (2.21b), is $\{q, S, R\}-$dissipative with $q = -1$, $S = \begin{bmatrix} 0 & 0 \end{bmatrix}$ and $R = (1.5)^2 I_2$.

A dissipative observer for the system (2.21), with parameters obtained from Rocha-Cózatl and Moreno, 2011, is given as

$$\begin{cases} \dot{v}_1 = v_2 - 10.2436(v_1 - x_1), \\ \dot{v}_2 = v_3 - 43.4485(v_1 - x_1), \\ \dot{v}_3 = \psi_1(x_1, v_2 - 3.8698(v_1 - x_1), v_3 - 24.3475(v_1 - x_1)) - 75.6773(v_1 - x_1). \end{cases} \tag{2.23}$$

Note that the dissipative observer can only ensure practical stability in the presence of bounded UP (see Figure 2.5).

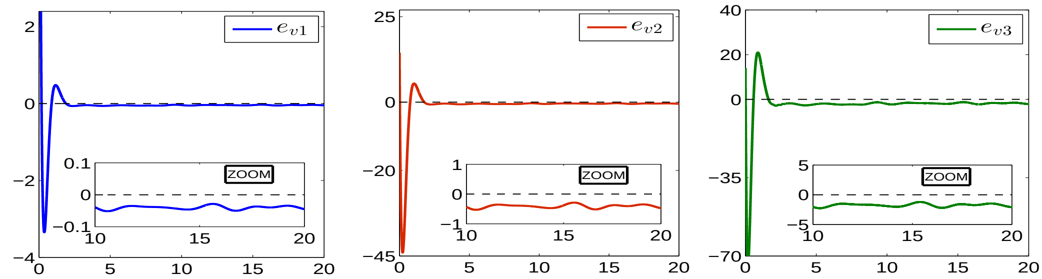

FIGURE 2.5: The estimation errors $e_{v1} = v_1 - x_1$, $e_{v2} = v_2 - x_2$, $e_{v3} = v_3 - x_3$ of the dissipative observer (2.23)

Sliding-mode observer. On the other hand, applying a second-order sliding-mode differentiator[2] (Levant, 2003) to the measured state x_1 of the system (2.21),

$$\begin{cases} \dot{z}_1 = -3L_f^{1/3}|z_1 - x_1|^{2/3} sign(z_1 - x_1) + z_2, \\ \dot{z}_2 = -1.5L_f^{1/2}|z_2 - \dot{z}_1|^{1/2} sign(z_2 - \dot{z}_1) + z_3, \\ \dot{z}_3 = -1.1L_f sign(z_3 - \dot{z}_2), \end{cases} \tag{2.24}$$

to estimate the states x_2 and x_3, one obtains the estimation error simulations (see Figure 2.6) with the parameter $L_f = 6$ and initial conditions $z_1(0) = z_2(0) = z_3(0) = 15$.

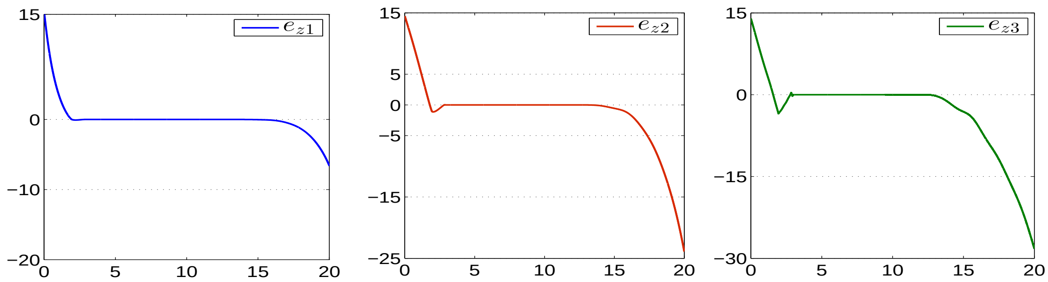

FIGURE 2.6: The estimation errors $e_{z1} = z_1 - x_1$, $e_{z2} = z_2 - x_2$, $e_{z3} = z_3 - x_3$ of the sliding-mode differentiator (2.24)

Figure 2.6 illustrates the loss of finite-time estimation of the sliding-mode differentiator. This happens because the system (2.21) does not have the BIBS property and consequently $\dot{x}_3$ is not bounded (see Figure 2.7). This drawback is also present when the robust exact uniformly convergent arbitrary order differentiator (Angulo, Moreno, and Fridman, 2013) and when the sliding-mode observer proposed by Floquet and Barbot, 2007 are implemented for this case.

This example illustrates the lack of effectiveness of both of the above mentioned approaches, when the system does not have the BIBS property and has relative degree higher than one w.r.t. the bounded UP.

Proposed observer. Consider again the system (2.21) where $\varrho_\psi = 2$. Find the parameters as follows

i) We chose $K = \begin{bmatrix} -1.39 & -1.288 & -0.892 \end{bmatrix}^T$, for which A_K is a Hurwitz matrix.

[2]Solutions of differential equations with discontinuous right-hand sides will be understood in Filippov's sense (Filippov, 1988).

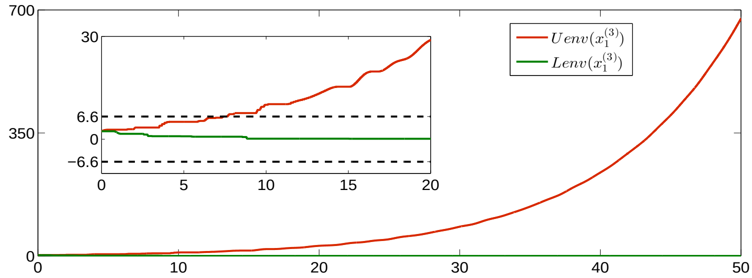

FIGURE 2.7: Upper envelope ($Uenv$) and Lower envelope ($Lenv$) of the signal $\dot{x}_3$

ii) We selected $Q = I_3$, $\tilde{N} = \begin{bmatrix} -10 & -80 \end{bmatrix}^T$ and $\epsilon = 0.093$.

iii) The matrix $P = \begin{bmatrix} 31.32 & -22.37 & 7.36 \\ -22.37 & 29.83 & -18.74 \\ 7.36 & -18.74 & 26.9 \end{bmatrix} > 0$ satisfies the Lyapunov inequality $PA_K + A_K^T P + \epsilon Q + \Lambda_{\varrho_\psi} < 0$, with

$$\Lambda_{\varrho_\psi} = \begin{bmatrix} \frac{\varrho_\psi}{9} & 0 & 0 \\ 0 & \frac{\varrho_\psi}{4} & 0 \\ 0 & 0 & \varrho_\psi \end{bmatrix}.$$

iv) With the parameter $l = 6$ the matrix inequality (2.15) is satisfied.

v) The orthogonal matrix $G = \begin{bmatrix} 0.49 & -0.65 & 0.58 \\ 0.75 & -0.04 & -0.66 \\ 0.45 & 0.76 & 0.47 \end{bmatrix}$ and the diagonal matrix $D = \begin{bmatrix} 3.88 & 0 & 0 \\ 0 & 1.49 & 0 \\ 0 & 0 & 62.68 \end{bmatrix}$, $\mathscr{O}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 8.34 & 1 & 0 \\ 46.368 & 8.34 & 1 \end{bmatrix}$ associated with the matrix P satisfies $P = G^T DG$.

vi) We calculated the parameter $L_f = 160246$ in (2.20) and the matrix $\mathscr{O}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 8.34 & 1 & 0 \\ 46.368 & 8.34 & 1 \end{bmatrix}$ in (2.19).

From the parameters found for the HOSM observer with SDS (2.17) one has the simulations (Figure 2.8) of the estimation error with initial conditions $v_i(0) = z_i(0) = 15$ for $i = 1, 2, 3$.

From the simulations (see Figure 2.8) it is clear that the proposed HOSM observer with SDS successfully achieves the exact estimation of the state.

It is well-known that L_f obtained in (2.20) can be a very crude estimation of the true value, so that in practice it is better to obtain its value by means of simulations. In the next section, a method allows an acceptable estimate of the parameters of the HOSM observer with SDS.

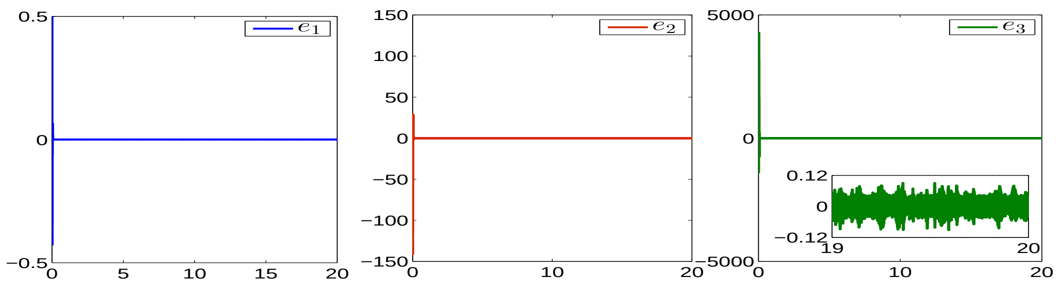

FIGURE 2.8: The estimation errors $e_1 = \hat{x}_1 - x_1$, $e_2 = \hat{x}_2 - x_2$, $e_3 = \hat{x}_3 - x_3$ of the proposed observer (2.17)

2.5.2 Proposed tuning method for observer gain design

Remember and define the following estimation errors:

$$e_{zy} = z - \begin{bmatrix} e_{vy} & \dot{e}_{vy} & \cdots & e_{vy}^{(n-1)} \end{bmatrix}^T, \qquad e_v = v - x, \qquad e_{vy} = Cv - y, \qquad e = \hat{x} - x. \tag{2.25}$$

Remark 1. *If the measured output error ($Cz - e_{vy}$) satisfies $Cz - e_{vy} = 0$, then the differentiator error satisfies $e_{zy} = 0$ and the estimation error of the HOSM observer with SDS ($e = \hat{x} - x$) satisfies $e = 0$.* ◇

Tuning rules. The effectiveness of the proposed observer is verified through the available signal $Cz - e_{vy}$.

i) Decrease the parameter L_f from the previously obtained and verify that $Cz - e_{vy} = 0$ holds,

ii) From a reasonable estimate of the gain L_f, decrease the parameter l in the SDS, verifying that it satisfies $Cz - e_{vy} = 0$.

It is also possible to reduce both parameters L_f and l simultaneously.

For the example (2.21), parameters L_f and l can be reduced to $L_f = 15$ and $l = 1$, where the proposed HOSM observer with SDS has the form

$$\begin{cases} \dot{v}_1 = v_2 - 1.39(v_1 - x_1), \\ \dot{v}_2 = v_3 - 1.288(v_1 - x_1), \\ \dot{v}_3 = \psi_1(x_1, v_2 - 10(v_1 - x_1), v_3 - 80(v_1 - x_1)) - 0.892(v_1 - x_1), \\ \dot{z}_1 = -2(2.47)\lceil z_1 - y + Cv \rfloor^{2/3} + z_2, \\ \dot{z}_2 = -1.5(3.87)\lceil z_2 - \dot{z}_1 \rfloor^{1/2} + z_3, \\ \dot{z}_3 = -1.1(15)\lceil z_3 - \dot{z}_2 \rfloor^{0}, \\ \hat{x}_1 = v_1 - z_1, \\ \hat{x}_2 = v_2 - 1.39 z_1 - z_2, \\ \hat{x}_3 = v_3 - 1.288 z_1 - 1.39 z_2 - z_3. \end{cases} \tag{2.26}$$

The simulations of Figures 2.9, 2.10, 2.11 are obtained with initial conditions $v_i(0) = z_i(0) = 15$ for $i = 1, 2, 3$. Figure 2.9 illustrates the estimation error of the SDS alone. Although its efficiency is not as good, the proposed HOSM observer with SDS ensures finite-time convergence to the true states, see Figure 2.10.

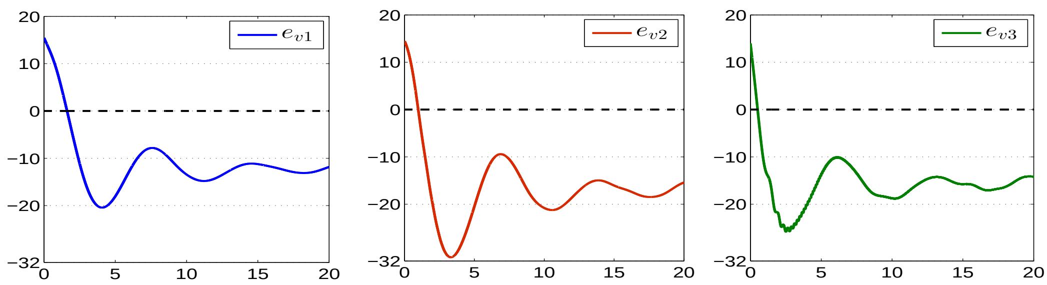

FIGURE 2.9: The estimation errors $e_{v1} = v_1 - x_1$, $e_{v2} = v_2 - x_2$, $e_{v3} = v_3 - x_3$ of the scaled dissipative stabiliser in (2.26)

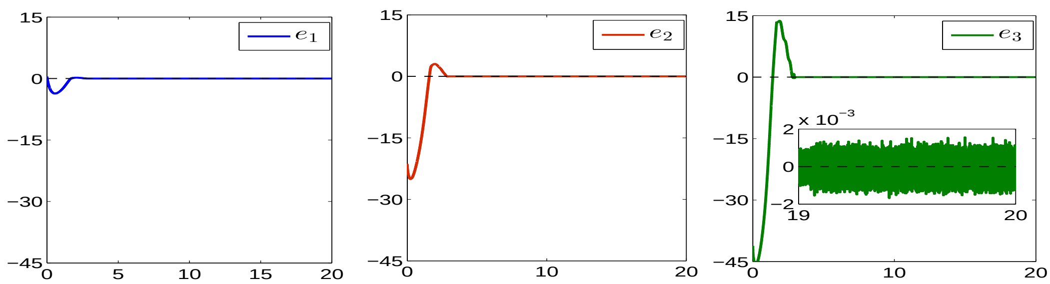

FIGURE 2.10: The estimation errors $e_1 = \hat{x}_1 - x_1$, $e_2 = \hat{x}_2 - x_2$, $e_3 = \hat{x}_3 - x_3$ of the proposed HOSM observer with SDS (2.26)

Remark 2. *It necessary to remark that the zoom in Figures 2.8 (right) and 2.10 (right) illustrates for a discrete realization the precision of the observer with gains selected according to the Theorem 3 is greater than the observer with tuned gains.* ◇

Figure 2.11 shows that the third derivative of the output error e_{vy} is bounded through its upper and lower envelope.

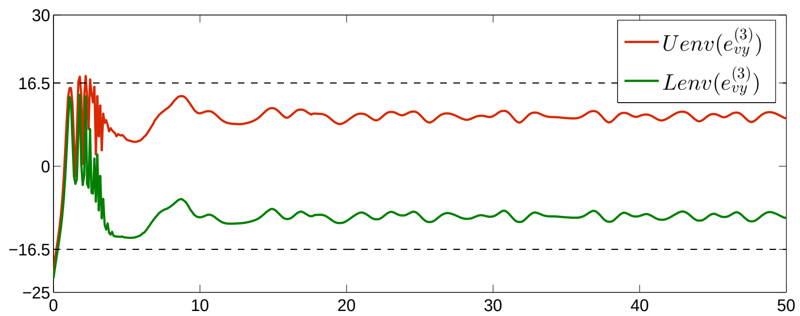

FIGURE 2.11: Upper envelope ($Uenv$) and Lower envelope ($Lenv$) of the third derivative of the output error $e_{vy}^{(3)}$

Table 2.1 helps to summarize the results of the simulations illustrating the observer properties.

	Dissipative observer Figure 2.5	**Sliding-Mode differentiator** Figure 2.6	**HOSM observer with SDS** Figure 2.8	**HOSM observer with SDS and gain tuning** Figure 2.10
Precision	Uniformly ultimately bounded	divergence after a finite time	Finite-time theoretically exact	Finite-time theoretically exact with better precision in discrete realization

TABLE 2.1

2.6 Conclusions

An observer that estimates globally, exactly and in finite time the unmeasured states, despite the presence of bounded UP, is designed for a class of chain of integrators which may not have the BIBS property. It is shown that the standard dissipative structure can be used for stabilization of the observation error, but in this case, the highest derivative of the output dissipative error depends on the dissipative observer gains and consequently it is not suitable to use it. A scaled dissipative stabiliser is proposed, ensuring that the highest derivative of the output estimation error is independent of the stabiliser gains. After this, a HOSM differentiator with the adjusted gains to the upper-bound of the unknown inputs is used. The properties of the suggested observer are illustrated by computer simulations.

Chapter 3

Dissipative approach to global sliding mode observers design for 1-DOF mechanical systems

3.1 Introduction

The control of mechanical systems requires information of both variables: position and velocity. Since usually only the position is measured, it is necessary to estimate the velocity by means of an observer. When the model of the system is nonlinear and the parameters, the inputs of the system are well known, there is extensive literature providing global and asymptotically converging velocity estimation, see e.g. (Gauthier, Hammouri, and Othman, 1992; Besançon, 2000; Besançon, 2007; Astolfi, Ortega, and Venkatraman, 2010). However, in the presence of uncertainty and perturbation (UP) (e.g. dry friction, unknown torque, etc.) the challenge of estimating globally, exactly the value of the velocity becomes more difficult, even more if finite-time convergence is required. If the perturbation in the system is *arbitrary*, the unknown input observer theory (Hautus, 1983; Rocha-Cózatl and Moreno, 2004; Rocha-Cózatl and Moreno, 2011) requires that the measured output has relative degree one with respect to (w.r.t.) the UP, but mechanical systems with UP have relative degree two w.r.t. the measured position. To allow the estimation of the velocity in this work one assumes that the UP is bounded.

For this purpose a discontinuous estimation algorithm is required, such as sliding-mode observers (Edwards, Spurgeon, and Tan, 2002; Barbot, Boukhobza, and Djemai, 2003; Spurgeon, 2008). One of their advantages is that they provide theoretically exact convergence to the true system's states, even in the presence of bounded perturbations and under the condition that the nonlinear system has a Bounded-Input-Bounded-State (BIBS) property w.r.t. the perturbations. Moreover, HOSM observers (Fridman et al., 2008; Barbot and Floquet, 2010; Bejarano, Pisano, and Usai, 2011; Pisano and Usai, 2011; Efimov et al., 2012) ensure this convergence in finite time.

In particular, for nonlinear mechanical systems with bounded UP the sliding-mode observers/differentiators (Levant, 1998; Davila, Fridman, and Levant, 2005; Xian et al., 2004; Moreno, 2009) require the system to be BIBS. To overcome this restriction (Apaza-Perez, Fridman, and Moreno, 2017) proposes a strategy connecting two observers in cascade: (i) A Luenberger observer ensuring that the estimation error converges to a neighborhood of zero; (ii) A higher-order sliding-mode differentiator that guarantees the global finite-time theoretically exact convergence to zero of the estimation error. However, this design strategy grows twice the order of the observer, and requires restrictive conditions for the gains design.

For observation of nonlinear systems that do not necessarily have the BIBS property, a dissipative approach is presented in Shim, Seo, and Teel, 2003; Moreno, 2004; Moreno, 2005 for systems without UP, and in Rocha-Cózatl and Moreno, 2004; Rocha-Cózatl and Moreno, 2011 with the presence of UP, results to be efficient. This technique contains as particular cases well-known observer design methods, e.g. Lipschitz (Rajamani, 1998) and high gain observers (Gauthier, Hammouri, and Othman, 1992). For systems with UP satisfying the conditions for existence of an observer, among them the relative-degree-one condition, the dissipative observer is able to estimate globally and exponentially the real states (Rocha-Cózatl and Moreno, 2004; Rocha-Cózatl and Moreno, 2011). But, when the relative-degree condition is not met but the UP is bounded the dissipative observer assures the convergence to a neighborhood of the origin of the estimation error (Moreno, 2005).

Summarizing, one can conclude that the observer design for mechanical systems with UP presents the following challenges: (i) relative degree two w.r.t. the UP; (ii) the system could be not BIBS, i.e. the sliding-mode differentiators cannot be used directly; (iii) the Coriolis term depending quadratically on the velocity; iv) there are uncertainties on the parameters of model, e.g. dry friction, hysteresis, etc; v) the system can be affected by external perturbations.

Main contribution. One-degree-of-freedom mechanical systems with Coriolis term, dry friction, bounded UP and other nonlinearities are considered. These systems may not have the BIBS property. For this class of systems, a global sliding-mode observer estimating the velocity theoretically exactly in finite time, is proposed.

The rest of the chapter is organized as follows. Section 3.2 contains a motivating example of the system class for which sliding-mode differentiators cannot ensure finite-time estimation. The problem statement is presented in Section 3.3. Section 3.4 presents a state transformation, to deal with the Coriolis term, and the proposed observer. The main results are presented in Section 3.5. Section 3.6 illustrates the main results with computer simulations. Section 3.7 provides some conclusions.

Notations. Throughout this paper we avail of the following notations: $\lceil\cdot\rfloor^p := |\cdot|^p sign(\cdot)$; $\lambda_M(D)$ and $\lambda_m(D)$ are the largest and the smallest eigenvalue of a square matrix D.

3.2 Motivation example

The following Lagrangian system was considered by Besançon, 2000

$$(1+\cos^2(q))\ddot{q} - \frac{1}{2}\sin(2q)\dot{q}^2 + g\sin(q) = \tau, \tag{3.1}$$

where $q \in \mathbb{R}$ is the position, $(1+\cos^2(q))$ is the inertia term, $-\frac{1}{2}\sin(2q)\dot{q}^2$ is the Coriolis force. Consider a more general system adding a continuous nonlinear term $-\frac{\sin^2(q)+1}{3}\dot{q}$, a discontinuous term (e.g. dry friction) $0.5\,\text{sign}(\dot{q})$ and a bounded perturbation $\tilde{\delta}(t)$ in the form:

$$(1+\cos^2(q))\ddot{q} - \frac{1}{2}\sin(2q)\dot{q}^2 + g\sin(q) - \frac{\sin^2(q)+1}{3}\dot{q} + 0.5\,\text{sign}(\dot{q}) = \tau + \tilde{\delta}(t). \tag{3.2}$$

This system has relative degree two w.r.t. the measured variable q and the UP.

Let's apply the generalized super-twisting (GST) algorithm (Moreno, 2011) as an observer to estimate the unmeasured variable in finite time

$$\begin{aligned} \dot{z}_1 &= -6.7\lceil e_1 \rfloor^{1/2} - 3.4\lceil e_1 \rfloor + z_2, \\ \dot{z}_2 &= -20\lceil e_1 \rfloor^{0} - 33\lceil e_1 \rfloor^{1/2} - 11\lceil e_1 \rfloor, \end{aligned} \tag{3.3}$$

where $e_1 = z_1 - q$ and the gains are obtained according to its methodology. For the simulations, consider the perturbation in the form

$$\tilde{\delta}(t) = 0.4\sin(3t)\cos(4t^3) + 0.5\cos(\pi t) + 0.6, \tag{3.4a}$$

$\tau = 0$ and the initial conditions as

$$(q(0), \dot{q}(0)) = (1, 1), \quad (z_1(0), z_2(0)) = (-20, -20). \tag{3.4b}$$

Figure 3.1:(a) illustrates that trajectories of system (3.2), with initial condition (3.4b), are not bounded. Figure 3.1:(a)-(b) illustrates that the differentiator state (3.3) converges at $t = 1.65\,[s]$ to the real state $\dot{q}$, but after $t = 10.2\,[s]$ the differentiator (3.3) loses convergence. This is because at this time the nonlinearity

$$\rho(q, \dot{q}) = \frac{\sin^2(q) + 1}{3(1 + \cos^2(q))}\dot{q} + \frac{\sin(2q)}{2(1 + \cos^2(q))}\dot{q}^2 - \frac{9.8\sin(q) + 0.5\,sign(\dot{q})}{1 + \cos^2(q)}$$

with perturbation $\tilde{\delta}(t)$ exceeds the value 20 corresponding to the gain of the discontinuous term in the differentiator (3.3), see Figure 3.1:(c).

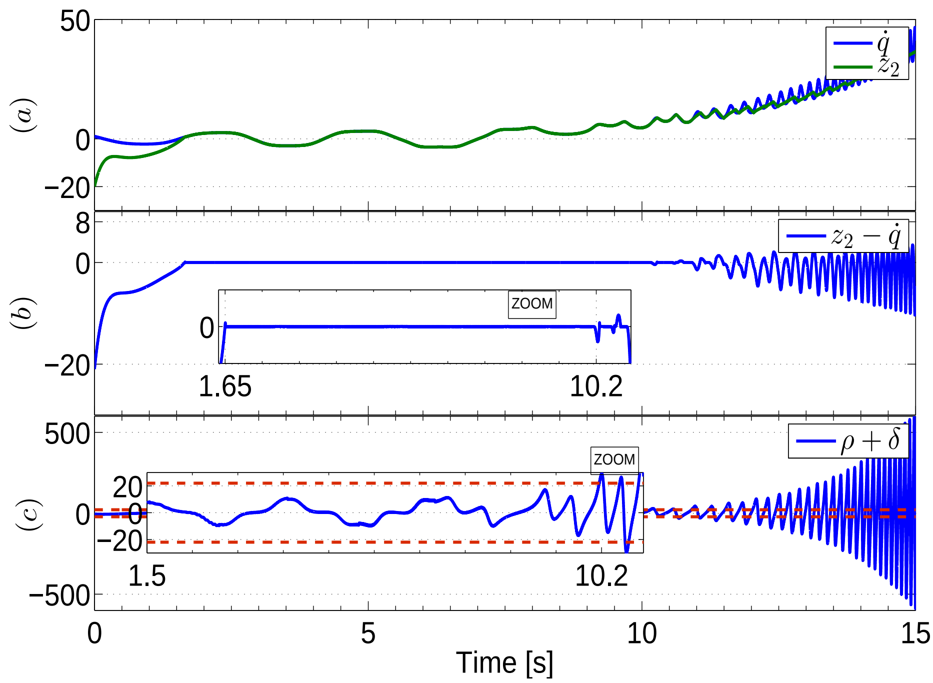

FIGURE 3.1: (a) The estimation state z_2 of differentiator (3.3) and the real state $\dot{q}$ of (3.2). (b) The estimation error $z_2 - \dot{q}$. (c) Nonlinearity and the perturbation $\rho + \tilde{\delta}$ overcome the gain 20

From this example one can conclude that for mechanical systems with UP and without BIBS property, the observer convergence is lost even if the GST algorithm based differentiator is applied. Hence, it is necessary to design an observer for systems not possessing the BIBS property.

3.3 Problem statement

Consider an one-degree-of-freedom (1-DOF) mechanical systems with UP given as

$$m(q)\ddot{q} + c(q)\dot{q}^2 + H(q,\dot{q}) + \eta \cdot \text{sign}(\dot{q}) + g(q) = \tau + \tilde{\delta}(t,q,\dot{q}) \tag{3.5}$$

where $q \in \mathbb{R}$ is the (measured) generalized position, $\dot{q}$ is the generalized velocity; $m(q)$ is the inertia term; $c(q)\dot{q}^2$ is the Coriolis and centrifugal force; $H(q,\dot{q})$ is a continuous nonlinearity (e.g. continuous frictions, air resistance, etc.); $\eta \in \mathbb{R}$ and $\eta \cdot sign(\dot{q})$ is the dry friction, which possibly contains relay terms depending on $\dot{q}$; $g(q)$ denotes gravitational forces; $\tilde{\delta}(t,q,\dot{q})$ contains UP; and τ is the measured torque.

Suppose that the family of 1-DOF mechanical systems with UP represented by (3.5) satisfies the following Assumptions:

A1. The inertia term $m(q)$ satisfies

$$\exists\ a_1, a_2 > 0\,; \forall q,\ a_1 \le m(q) \le a_2, \tag{3.6}$$

$$\dot{m}(q) = 2c(q)\dot{q}. \tag{3.7}$$

A2. The UP term $\tilde{\delta}(t,q,\dot{q})$ is bounded, i.e. there exists a constant $L_{\tilde{\delta}}$ such that

$$|\tilde{\delta}(t,q,\dot{q})| \le L_{\tilde{\delta}}.$$

A3. The nonlinearity $H(q,\dot{q})$ can be represented as

$$H(q,\dot{q}) = h_1(q,\dot{q}) + h_2(q,\dot{q}), \quad \text{such that ,}$$

where

i) for the function $\varphi(q,\dot{q}) := -\frac{h_1(q,\dot{q})}{m(q)}$ there exist $\{p,s,r\}$ with $p < 0$ such that the nonlinearity

$$\Phi(v_1,v_2,v_3) := \varphi(v_1, v_2+v_3) - \varphi(v_1,v_2),$$

is $\{p,s,r\}$-dissipative, i.e.

$$\begin{bmatrix} \Phi(v_1,v_2,v_3) \\ v_3 \end{bmatrix}^T \begin{bmatrix} p & s \\ s & r \end{bmatrix} \begin{bmatrix} \Phi(v_1,v_2,v_3) \\ v_3 \end{bmatrix} \ge 0,$$

for all v_1, v_2, $v_3 \in \mathbb{R}$.

ii) for the function $\psi(q,\dot{q}) := -\frac{h_2(q,\dot{q})}{m(q)}$ there exist $\{0,\bar{s},\bar{r}\}$ with $\bar{s} < 0$ such that the nonlinearity

$$\Psi(v_1,v_2,v_3) := \psi(v_1, v_2+v_3) - \psi(v_1,v_2),$$

is $\{0,\bar{s},\bar{r}\}$-dissipative, i.e.

$$\begin{bmatrix} \Psi(v_1,v_2,v_3) \\ v_3 \end{bmatrix}^T \begin{bmatrix} 0 & \bar{s} \\ \bar{s} & \bar{r} \end{bmatrix} \begin{bmatrix} \Psi(v_1,v_2,v_3) \\ v_3 \end{bmatrix} \ge 0,$$

for all v_1, v_2, $v_3 \in \mathbb{R}$.

Remark 3. *Assumption (3.7) is a standard condition for inertia and Coriolis terms in mechanical systems (Besançon, 2000; Spong, Hutchinson, and Vidyasagar, 2006). The decomposition of H, for Assumption A3, represents two classes of nonlinearities: the nonlinearity h_1 contains globally Lipschitz functions w.r.t. the velocity (e.g. viscous friction) and the nonlinearity h_2 contains monotone functions w.r.t. the velocity that do not need to be globally Lipschitz (e.g. air resistance $-k\lceil \dot{q} \rfloor^2$ or the family $-k\lceil \dot{q} \rfloor^\alpha$ with $\alpha > 0$).*

Considering $\xi_1 = q$, $\xi_2 = \dot{q}$, the state space representation of (3.5) is given by

$$\begin{aligned} \dot{\xi}_1 &= \xi_2, \\ \dot{\xi}_2 &= \varphi(\xi_1, \xi_2) + \psi(\xi_1, \xi_2) + \alpha(\xi_1)\xi_2^2 + \vartheta(\xi_1) + u + w, \\ y &= \xi_1, \end{aligned} \tag{3.8}$$

where

$$\varphi(\xi_1, \xi_2) = -\frac{h_1(\xi_1, \xi_2)}{m(\xi_1)}, \quad \psi(\xi_1, \xi_2) = -\frac{h_2(\xi_1, \xi_2)}{m(\xi_1)},$$

$$\alpha(\xi_1) = \frac{-c(\xi_1)}{m(\xi_1)}, \quad \vartheta(\xi_1) = -\frac{g(\xi_1)}{m(\xi_1)}, \quad u = \frac{\tau}{m(\xi_1)},$$

$$w = \frac{\eta \cdot sign(\xi_2)}{m(\xi_1)} + \frac{\tilde{\delta}(t, \xi_1, \xi_2)}{m(\xi_1)}.$$

Notice that $|w| \le L_w$ with $L_w = \frac{\eta + L_{\tilde{\delta}}}{a_1}$. We assume that the solutions of the system (3.8) are defined in the sense of Filippov, 1988 and they exist for all $t \ge 0$.

The main goal of this chapter is to design a global observer for system (3.8) estimating the unmeasured velocity $\dot{q}$ globally, theoretically exactly and in finite time, without assuming BIBS property for (3.8).

3.4 Construction of the observer

3.4.1 Transformation to deal with the Coriolis force

Consider the function

$$\Upsilon(y) := \exp\left(-\int_a^y \alpha(\mu) d\mu \right), \tag{3.9}$$

where $\alpha(\mu) = -\frac{c(\mu)}{m(\mu)}$ and "a" is a constant defined in the domain of $\alpha(\cdot)$. From the condition (3.7) follows $\Upsilon(y) = \sqrt{\frac{m(y)}{m(a)}}$ and its multiplicative inverse as $(\Upsilon(y))^{-1} = \sqrt{\frac{m(a)}{m(y)}}$.

Assumption A1 implies

$$0 < d_1 \le \Upsilon(y) \le d_2, \tag{3.10}$$

where $d_1 = \sqrt{\frac{a_1}{m(a)}}$ and $d_2 = \sqrt{\frac{a_2}{m(a)}}$.

Function $T_1(\xi_1) := \int_a^{\xi_1} \Upsilon(\mu) d\mu$ is a diffeomorphism, since it is differentiable and, due to (9), it is monotonically increasing, invertible and the inverse $T_1^{-1}(\cdot)$ is differentiable. One can define a diffeomorphism $T(\cdot, \cdot)$ as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} T_1(\xi_1) \\ T_2(\xi_1, \xi_2) \end{bmatrix} = T(\xi_1, \xi_2) := \begin{bmatrix} \int_a^{\xi_1} \Upsilon(\mu) d\mu \\ \Upsilon(\xi_1) \cdot \xi_2 \end{bmatrix}, \tag{3.11}$$

and T^{-1} is given by

$$T^{-1}(x_1, x_2) = \begin{bmatrix} T_1^{-1}(x_1) \\ (\Upsilon(T_1^{-1}(x_1)))^{-1} \cdot x_2 \end{bmatrix}. \tag{3.12}$$

This transformation is similar to one used by Krener and Respondek, 1985.

Using the transformation (3.11) we obtain the following transformed system from (3.8)

$$\begin{aligned} \dot{x}_1 =& x_2, \\ \dot{x}_2 =& \Upsilon(\tilde{T}_1)\left(\varphi(\tilde{T}_1, (\Upsilon(\tilde{T}_1))^{-1}x_2) + u + w(t, \xi_1, \xi_2)\right) + \\ & + \Upsilon(\tilde{T}_1)(\psi(\tilde{T}_1, (\Upsilon(\tilde{T}_1))^{-1}x_2) + \vartheta(\tilde{T}_1)), \end{aligned} \tag{3.13}$$

where $\tilde{T}_1 := T_1^{-1}(x_1)$ and x_1 is the measured variable. Notice that the transformed system (3.13) does not contain the quadratic term of the Coriolis force.

3.4.2 Observer structure

For system (3.13), we propose the following observer

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 - k_1\phi_1(\hat{x}_1 - x_1), \\ \dot{\hat{x}}_2 &= \Upsilon(\tilde{T}_1)\,\varphi\left(\tilde{T}_1, (\Upsilon(\tilde{T}_1))^{-1}\left(\hat{x}_2 + k_4\phi_1(\hat{x}_1 - x_1)\right)\right) + \\ &\quad + \Upsilon(\tilde{T}_1)\,\psi\left(\tilde{T}_1, (\Upsilon(\tilde{T}_1))^{-1}\left(\hat{x}_2 + k_3\phi_1(\hat{x}_1 - x_1)\right)\right) + \\ &\quad + \Upsilon(\tilde{T}_1)\left(u + \vartheta(\tilde{T}_1)\right) - k_2\phi_2(\hat{x}_1 - x_1), \end{aligned} \tag{3.14a}$$

where $\tilde{T}_1 = T_1^{-1}(x_1)$, the nonlinearities ϕ_1 and ϕ_2 as

$$\begin{aligned} \phi_1(\cdot) &:= \mu_1\lceil\cdot\rfloor^{1/2} + \mu_2\lceil\cdot\rfloor, \\ \phi_2(\cdot) &:= \frac{\mu_1^2}{2}\lceil\cdot\rfloor^0 + \frac{3\mu_1\mu_2}{2}\lceil\cdot\rfloor^{1/2} + \mu_2^2\lceil\cdot\rfloor. \end{aligned} \tag{3.14b}$$

where the gains to design for this system are

$$k_1, k_2, \mu_1, \mu_2 > 0; \quad k_3, k_4 \in \mathbb{R}. \tag{3.14c}$$

This observer is a copy of the transformed system (3.13) with nonlinear injection terms ϕ_1 and ϕ_2. These injections appear additively in the observer and also within the nonlinearities φ and ψ. The discontinuous term $\lceil\cdot\rfloor^0$ in ϕ_2 ensures robustness of the observer against bounded UP, and the other nonlinear terms ensure finite-time convergence to the real states.

The estimated state in original coordinates is given by

$$\hat{\xi}_2 = (\Upsilon(\tilde{T}_1))^{-1}\hat{x}_2, \tag{3.14d}$$

and the observer dynamics is

$$\dot{\hat{\xi}}_1 = \hat{\xi}_2 - \frac{k_1}{\Upsilon(\hat{\xi}_1)}\phi_1(\mathscr{T}), \tag{3.15}$$

$$\dot{\hat{\xi}}_2 = \frac{\Upsilon(\xi_1)}{\Upsilon(\hat{\xi}_1)}\left[\varphi\left(\xi_1, \frac{\Upsilon(\hat{\xi}_1)}{\Upsilon(\xi_1)}\hat{\xi}_2 + \frac{k_4}{\Upsilon(\xi_1)}\phi_1(\mathscr{T})\right) + u\right] +$$
$$+ \frac{\Upsilon(\xi_1)}{\Upsilon(\hat{\xi}_1)}\left[\psi\left(\xi_1, \frac{\Upsilon(\hat{\xi}_1)}{\Upsilon(\xi_1)}\hat{\xi}_2 + \frac{k_3}{\Upsilon(\xi_1)}\phi_1(\mathscr{T})\right) + \vartheta(\tilde{T}_1)\right] +$$
$$+ \alpha(\xi_1)\left(\hat{\xi}_2^2 - \frac{k_1}{\Upsilon(\hat{\xi}_1)}\phi_1(\mathscr{T})\hat{\xi}\right) - \frac{k_2}{\Upsilon(\hat{\xi}_1)}\phi_2(\mathscr{T}),$$

where $\mathscr{T} = \mathscr{T}(\hat{\xi}) - \mathscr{T}(\xi)$.

3.5 Main results

Theorem 4. *If Assumptions A1, A2 and A3 are satisfied, then there exists a set of gains (3.14c) such that the system (3.15) is an observer converging globally theoretically exact in finite time to the states of system (3.8).* ▲

The proof of Theorem 4 is in Appendix B.2.

Design of observer gains. The gains k_1, k_2, k_3, k_4, μ_1, μ_2 of the observer (3.15)-(3.14b) can be chosen using the matrix inequality (3.16).

Lemma 2. *For given constants L_w, d_2, p, s, r, $\bar{s}$, $\bar{r}$ with $L_w \geq 0$, $d_2 > 0$, and $p, \bar{s} < 0$, there exist constants $p_1, \epsilon, \theta_3, k_i, \mu_i, \theta_i > 0$, $i = 1, 2$, $k_4 < 0$ and k_3 such that the following matrix inequality*[1]

$$\begin{bmatrix} P_A + \tilde{\theta}_1 H_E + \theta_2 H_C + \tilde{\theta}_3 \bar{H}_E + \epsilon I & \star & \star \\ B^T P & \theta_1 p & \star \\ B^T P & 0 & -\theta_2\left(\frac{\mu_1^2}{2d_2}\right)^2 \end{bmatrix} \leq 0, \tag{3.16}$$

is satisfied, where $A = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix}$, $P = \begin{bmatrix} p_1 & -\theta_3\bar{s}k_4 \\ -\theta_3\bar{s}k_4 & -\theta_3\bar{s} \end{bmatrix} > 0$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $P_A = A^T P + PA$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $E = \begin{bmatrix} k_3 & 1 \end{bmatrix}$, $\bar{E} = \begin{bmatrix} k_4 & 1 \end{bmatrix}$, $H_E = E^T\left(\frac{|s|+\sqrt{-pr+s^2}}{\sqrt{-p}}\right)^2 E$, $\bar{H}_E = \bar{E}^T \frac{d_2\bar{r}(sign(\bar{r})+1)}{2}\bar{E}$, $H_C = C^T (L_w)^2 C$, $\tilde{\theta}_1 = \frac{\theta_1}{\mu_2^2}$ *and* $\tilde{\theta}_3 = \frac{\theta_3}{\mu_2}$. ▲

The proofs of Lemma 2 is in Appendix B.1.

[1] The symbol $\star$ indicates that matrix is symmetric.

3.6 Simulation Examples

3.6.1 Motivational example of Chapter 1

Returning to example (1.2) given in Section 1.1.3; this system (1.2) is identified by system (3.5) with $m(q) = 1$, $c(q) = 0$, $h_1(q, \dot{q}) = -\frac{\sin^2(0.5q)}{2(1+\cos^2(q))}\dot{q}$, $h_2(q, \dot{q}) = 0$, $g(q) = \frac{9.8b\sin(0.2q)}{4(1+\cos^2(q))}$, $\eta = 0$, $\tau = 0$. For this example, Assumptions A1, A2 and A3-i) are satisfied, with $a_1 = a_2 = 1$, $p = -1$, $s = 0$, $r = \frac{2}{3}$.

The function $\Upsilon(y)$ in (3.9) for this example is given as $\Upsilon(y) = 1$, which satisfies (3.10) with $a = 0$, $d_1 = d_2 = 1$. the state transformation $T(\xi_1, \xi_2)$ in (3.11) is the identity function. Then from (3.16) one gets $k_1 = -2.16$, $k_2 = -3.65$, $k_3 = 0.01$, $\epsilon = 0.01$ $\theta_1 = 2.35$, $\mu_1 = 1$, $\mu_2 = 1.15$, $P = \begin{bmatrix} 2.48 & -1.19 \\ -1.19 & 1.12 \end{bmatrix}$. For the simulations the initial conditions were considered as $(x_1(0), x_2(0)) = (-0.5, 6.3)$, $(\hat{x}_1(0), \hat{x}_2(0)) = (3, 3)$ and the UP is the same as in (1.4).

The estimated states $\hat{x}_1$ and $\hat{x}_2$ converge to real states x_1 and x_2 from $t = 0.20\,[s]$ as it is illustrated in Figure 3.2. Figure 3.3 illustrates the estimation error.

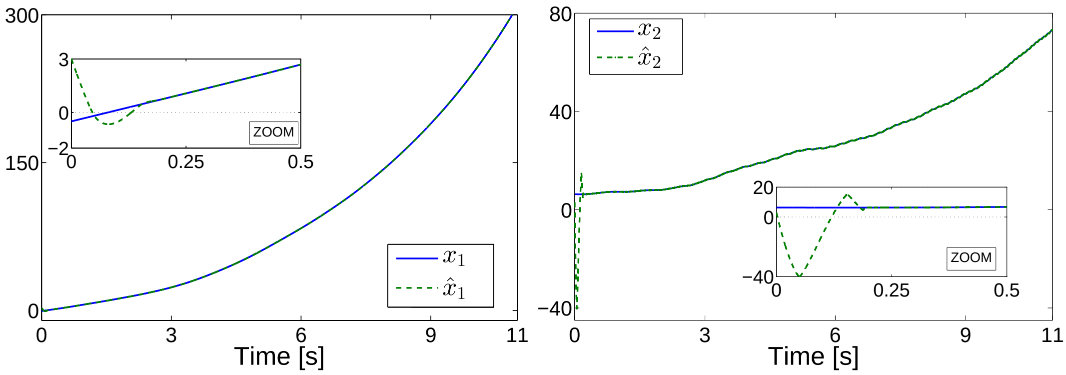

FIGURE 3.2: The estimation states $\hat{x}_1$, $\hat{x}_2$ in the observer and real states x_1, x_2 of (1.2)

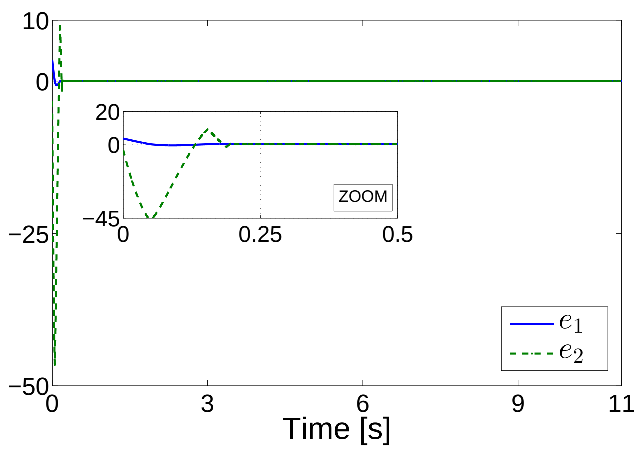

FIGURE 3.3: The estimation error $e_1 = \hat{x}_1 - x_1$ and $e_2 = \hat{x}_2 - x_2$

3.6.2 Motivational example of Section 3.2.

Let us return to the motivational example (3.2). For this example, Assumptions A1, A2 and A3-i) are satisfied, with $m(q) = 1 + \cos^2(q)$, $c(q) = -\frac{1}{2}\sin(q)$, $\varphi(q,\dot{q}) = \frac{\sin^2(q)+1}{3(1+\cos^2(q))}\dot{q}$ and parameters $p = -1$, $s = 0$, $r = \frac{2}{3}$.

The function $\Upsilon(y)$ in (3.9) for this example is given as $\Upsilon(y) = \sqrt{1+\cos^2(y)}$ which satisfies (3.10) with $d_1 = 1$, $d_2 = \sqrt{2}$. Fix $a = \frac{\pi}{2}$ for transformation (3.11). Then from (15) one gets $k_1 = -2.16$, $k_2 = -3.65$, $k_3 = 0.01$, $\epsilon = 0.01$ $\theta_1 = 2.35$, $\mu_1 = 1$, $\mu_2 = 1.15$, $P = \begin{bmatrix} 2.48 & -1.19 \\ -1.19 & 1.12 \end{bmatrix}$. For the simulations the initial conditions and the perturbation are the same as in (3.4).

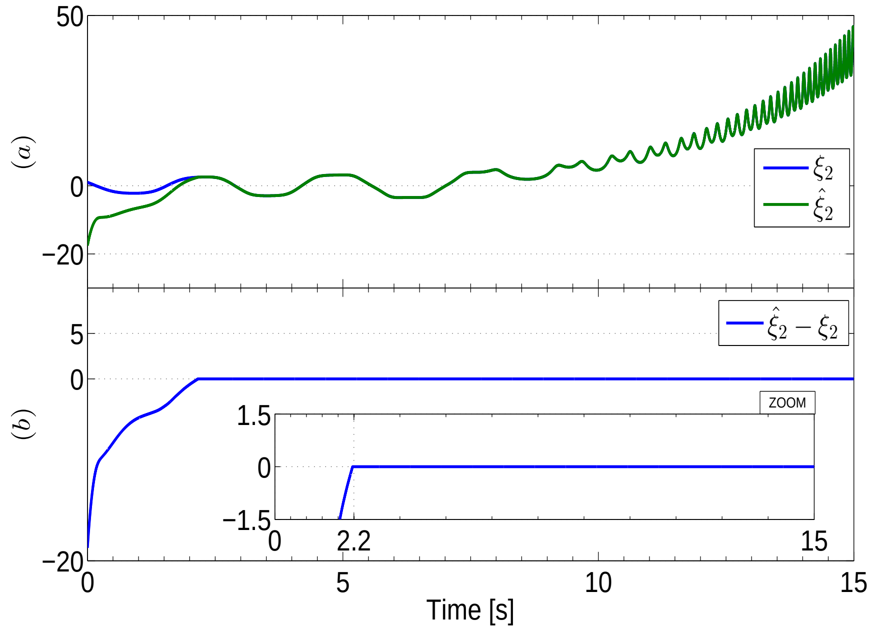

FIGURE 3.4: (a) The real states ξ_2 on (3.2) and their estimation state $\hat{\xi}_2$. (b) The estimation error $\hat{\xi}_2 - \xi_2$

Figure 3.4 illustrates the convergence for all $t > 2.2[s]$ and it is not lost as in the motivational example (3.2).

3.6.3 Example

Consider the following system, including nonlinearities with properties as in Assumption A3,

$$\ddot{q} + \dot{q}^2 sign(\dot{q}) - (\cos^2(q) + 1)\dot{q} = \tau + \tilde{\delta}(t). \tag{3.17}$$

Since the inertia term $m(q) = 1$ is a constant, there is no Coriolis term, i.e. $c(q) = 0$, and Assumption A1 is met. The nonlinearity $\varphi = (\cos^2(q)+1)\dot{q}$ satisfies Assumption A3-i) with parameters $p = -1$, $s = 0$, $r = 2$, while the nonlinearity $\psi = -\dot{q}^2 sign(\dot{q})$ fulfills A3-ii) with parameters $\bar{s} = -\frac{1}{2}$, $\bar{r} = 0$.

From the matrix inequality (3.16) one gets $k_1 = 2.41$, $k_2 = 6.77$, $k_3 = -1$, $k_4 = -1.28$, $\epsilon = 440.28$, $\theta_1 = 1231.5$, $\theta_2 = 1$, $\theta_3 = 794.78$, $\mu_1 = 8.77$, $\mu_2 = 4$, $p_1 = 1793.64$. For the simulations the initial conditions and the perturbation are the same as in (3.4). Figure 3.5 illustrates that the convergence in finite time is ensured for all $t > 0.5\,[s]$.

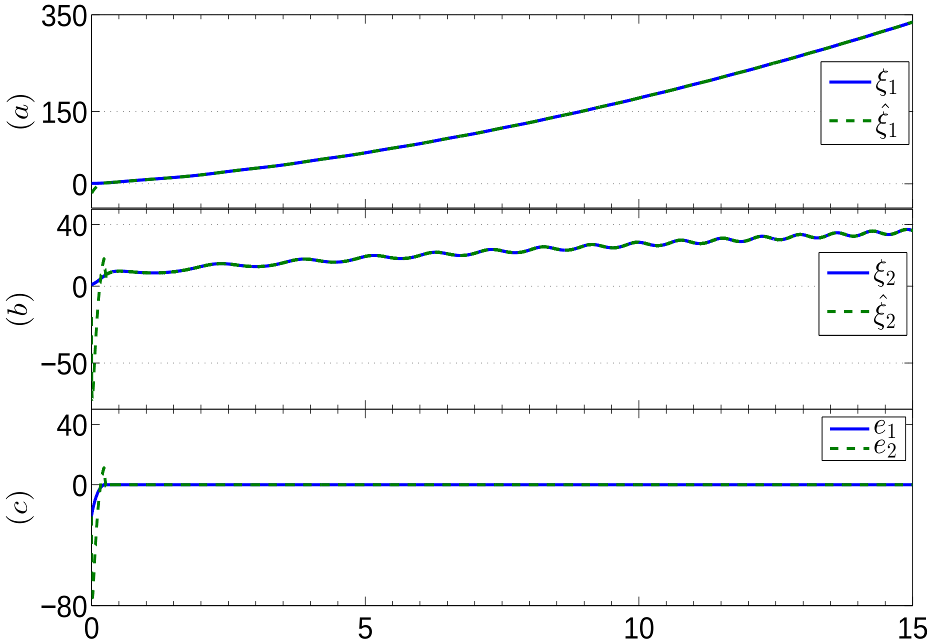

FIGURE 3.5: (a)-(b) The real states ξ_1, ξ_2 and their estimation states $\hat{\xi}_1$, $\hat{\xi}_2$, respectively. (c) The estimation errors $e_1 = \hat{\xi}_1 - \xi_1$, $e_2 = \hat{\xi}_2 - \xi_2$

3.7 Conclusions

The class of mechanical systems (3.5) with the Coriolis term, dry friction, UP, that are not BIBS, was considered. The state transformation (3.11) was introduced to deal with the Coriolis term. For these systems, the global sliding-mode observer (3.15) with theoretically exact finite-time convergence using dissipative properties, was proposed. The gains of this observer are obtained from a feasible matrix inequality (3.16).

Chapter 4

Dissipative approach to global sliding mode observers design for 2-DOF mechanical systems

4.1 Introduction

Control of mechanical systems usually requires the information about position and velocity. Often only the position is available, that is why the observer is needed. One challenge in building observers to estimate the velocity in mechanical systems is the presence of uncertainty/perturbation (UP). The sliding-mode (SM) observers/differentiators (Levant, 1998; Xian et al., 2004; Davila, Fridman, and Levant, 2005; Fridman et al., 2008; Moreno, 2009; Barbot and Floquet, 2010; Pisano and Usai, 2011; Efimov et al., 2012) provide theoretically exact finite-time convergence to the real system's states despite the presence of bounded UP when the system has the bounded-input-bounded-state (BIBS) property, and consequently the convergence is semi-global. For systems without the BIBS property, the presence of Coriolis and centrifugal forces create even more problems in the observer design (Levant, 1998; Xian et al., 2004; Davila, Fridman, and Levant, 2005; Moreno, 2009) due to their quadratic terms on velocities.

In this work, a class of two-degree-of-freedom (2-DOF) mechanical systems with bounded UP, which could not have the BIBS property is considered.

State of art and motivation: $1)$ The dissipative observers (Rocha-Cózatl and Moreno, 2004; Rocha-Cózatl and Moreno, 2011) result to be efficient to deal with the BIBS restriction. If the systems with UP satisfy the conditions for the existence of an observer, among them the relative degree one condition, then they are able to estimate globally and exponentially the real states using dissipative properties which could have the nonlinearities. But mechanical systems with UP have relative degree two w.r.t. the measured position.

$2)$ There are many works (Besançon, 2000; Mabrouk, Mazenc, and Vivalda, 2004; Astolfi, Ortega, and Venkatraman, 2010; Stamnes, Aamo, and Kaasa, 2011) dealing with the Coriolis and centrifugal forces, which provide global observers when the model of the mechanical system is completely known. Through state transformations, Astolfi, Ortega, and Venkatraman, 2010; Stamnes, Aamo, and Kaasa, 2011 propose observers with fairly high dimension, namely $3n+1$ and $2n+2$ respectively, where n is the dimension of the unmeasured velocity. Besançon, 2000; Mabrouk, Mazenc, and Vivalda, 2004 propose observers with the same dimension of the system for a class of mechanical systems. However, the challenge of dealing with viscous and dry frictions, UP and obtaining an estimation theoretically exact of velocity was not considered.

3) To overcome the BIBS restriction and the presence of Coriolis and centrifugal forces, in Apaza-Perez, Moreno, and Fridman, 2016 a global observer for 1-DOF mechanical systems with UP is proposed. The state transformation used in Apaza-Perez, Moreno, and Fridman, 2016 is not applicable to 2-DOF mechanical systems.

Main contribution. We consider 2-DOF mechanical systems with Coriolis and centrifugal forces and bounded UP, which may not have BIBS property. For this class of systems, a dissipativity based sliding-mode observer, is proposed. The observer gains can be obtained from feasible matrix inequalities.

The rest of the chapter is organized as follows. The problem statement is presented in Section 4.2. Section 4.3 presents the construction of the proposed observer. The main results are shown in Section 4.4. Section 4.5 illustrates the proposed observer effectiveness through simulation results using the cart-pendulum system. All proofs are in the Appendix.

4.2 Problem statement

Consider the 2-DOF mechanical system with UP:

$$M(x)\ddot{x} + C(x,\dot{x})\dot{x} + G(x) + \psi(x,\dot{x}) + H\dot{x} + \Lambda sign(\dot{x}) = Du + \tilde{\delta}(t,x,\dot{x}), \tag{4.1}$$

where $x = (x_1, x_2)^T \in \mathbb{R}^2$ is the measured position, $M(x) \in \mathbb{R}^{2\times 2}$ is the inertia matrix, $C(x,\dot{x})\dot{x}$ represents Coriolis and centrifugal forces, $\psi = (\psi_1, \psi_2)^T$ is a continuous nonlinearity (e.g. other types of frictions, air resistance, etc.), $H, \Lambda, D \in \mathbb{R}^{2\times 2}$, $H\dot{x}$ and $\Lambda sign(\dot{x})$ are viscous and dry frictions, $G(x)$ denotes gravitational forces, $\tilde{\delta}(t,x,\dot{x})$ contains perturbations, and $u \in \mathbb{R}^2$ is the control input.

The objective is to design an observer with theoretically exact global finite-time convergence to the real values of the velocity.

In the family of 2-DOF systems (4.1), the entries of the Coriolis and centrifugal matrix $C(x,\dot{x}) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ is defined from the entries of $M(x)$ through the Christoffel symbols (Spong, Hutchinson, and Vidyasagar, 2006) as

$$c_{kj} = \frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial m_{kj}(x)}{\partial x_i} + \frac{\partial m_{ki}(x)}{\partial x_j} - \frac{\partial m_{ij}(x)}{\partial x_k}\right)\dot{x}_i, \tag{4.2}$$

for $k, j = 1, 2$.

Consider the family of systems (4.1), which additionally satisfies the following assumptions:

P-1 The matrix $M(x)$ depends only on one variable x_2 as

$$M(x_2) = \begin{bmatrix} m_{11} & m_{12}(x_2) \\ m_{12}(x_2) & m_{22}(x_2) \end{bmatrix}.$$

P-2 There exist two constants $\alpha_1 > 0$, $\alpha_2 > 0$ such that

$$0 < \alpha_1 I \leq M(x_2) \leq \alpha_2 I, \text{ for all } x_2, \tag{4.3}$$

is satisfied, where I denotes the identity matrix of dimension 2×2.

P-3 The terms $(H\Upsilon_{x_2})_{11}$ and $m_{11}(H\Upsilon_{x_2})_{22} - m_{12}(x_2)(H\Upsilon_{x_2})_{12}$ are non negative, where $\Upsilon_{x_2} = \begin{bmatrix} \frac{1}{m_{11}} & -\frac{m_{12}(x_2)}{m_{11}\alpha(x_2)} \\ 0 & \frac{1}{\alpha(x_2)} \end{bmatrix}$ and $\alpha(x_2) = \sqrt{\frac{det(M(x_2))}{m_{11}}}$.

P-4 The perturbation/uncertainty $\tilde{\delta}(t, x, \dot{x})$ is bounded, i.e. there exists a constant $L_{\tilde{\delta}} > 0$ such that $\|\tilde{\delta}(t, x, \dot{x})\| \leq L_{\tilde{\delta}}$.

P-5 There exist $Q, S, R \in \mathbb{R}^{2\times 2}$, with $Q < 0$ such that the nonlinearity

$$\Psi(\xi_1, \xi_2, h) := \psi(\xi_1, \xi_2) - \psi(\xi_1, \xi_2 + h),$$

is $\{Q, S, R\}$-dissipative, i.e.

$$\begin{bmatrix} \Psi(\xi_1, \xi_2, h) \\ h \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \Psi(\xi_1, \xi_2, h) \\ h \end{bmatrix} \geq 0,$$

for all $\xi_1, \xi_2, h \in \mathbb{R}^2$.

Using the relationships from the Christoffel symbols (4.2) and Assumption P-1, the Coriolis and centrifuges matrix is reduced to

$$C(x, \dot{x}) = \begin{bmatrix} 0 & m'_{12}(x_2)\dot{x}_2 \\ 0 & \frac{1}{2}m'_{22}(x_2)\dot{x}_2 \end{bmatrix}. \tag{4.4}$$

4.3 Observer design

Transformation of states to deal with Coriolis term. If the system (4.1) satisfies the assumptions P1 and P2, then with the notations introduced above and setting $v = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T = Du - G(x)$, $\delta = \begin{bmatrix} \delta_1 & \delta_2 \end{bmatrix}^T = \tilde{\delta} - \Lambda sign(\dot{x})$ system (4.1) is expressed as

$$\begin{cases} \dot{x} = z, \\ \dot{z} = M^{-1}(x)\left[v - C(x, z)z - Hz - \psi(x, z) + \delta(t, x, z)\right]. \end{cases} \tag{4.5}$$

Consider the diffeomorphism (state transformation)

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} x_1 + \int_0^{x_2} \frac{m_{12}(s)}{m_{11}} ds \\ x_2 \\ m_{11}z_1 + m_{12}(x_2)z_2 \\ \alpha(x_2)z_2 \end{bmatrix}, \tag{4.6}$$

where its inverse is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \theta_1 - \int_0^{\theta_2} \frac{m_{12}(s)}{m_{11}} ds \\ \theta_2 \\ \frac{1}{m_{11}}\left(w_1 - m_{12}(\theta_2)\alpha(\theta_2)^{-1}w_2\right) \\ \alpha(\theta_2)^{-1}w_2 \end{bmatrix}, \tag{4.7}$$

where $\alpha(x_2) = \sqrt{\frac{det(M(x_2))}{m_{11}}}$, and notice that $\alpha(x_2) \leq \sqrt{\alpha_2}$ is satisfied. The transformation (4.6) is inspired by one proposed in Mabrouk, Mazenc, and Vivalda, 2004 for the case $H = 0$, $\psi(x, z) = 0$, $\tilde{\delta} = 0$, $\Lambda = 0$.

The transformed system from (4.5) using (4.6) is given as

$$\begin{aligned}
\dot{\theta}_1 &= \frac{w_1}{m_{11}},\\
\dot{w}_1 &= -(H\Upsilon_{\theta_2})_{11}w_1 - (H\Upsilon_{\theta_2})_{12}w_2 - \psi_1(x,\Upsilon_{\theta_2}w) + v_1 + \delta_1,\\
\dot{\theta}_2 &= \frac{w_2}{\alpha(\theta_2)},\\
\dot{w}_2 &= \frac{m_{11}[-(H\Upsilon_{\theta_2})_{21}w_1 - (H\Upsilon_{\theta_2})_{22}w_2]}{m_{11}\alpha(\theta_2)}+\\
&\quad - \frac{m_{12}(\theta_2)[-(H\Upsilon_{\theta_2})_{11}w_1 - (H\Upsilon_{\theta_2})_{12}w_2]}{m_{11}\alpha(\theta_2)}+\\
&\quad + \frac{m_{11}(-\psi_2(x,\Upsilon_{\theta_2}w)) - m_{12}(\theta_2)(-\psi_1(x,\Upsilon_{\theta_2}w))}{m_{11}\alpha(\theta_2)}\\
&\quad + \frac{m_{11}v_2 - m_{12}(\theta_2)v_1 + m_{11}\delta_2 - m_{12}(\theta_2)\delta_1}{m_{11}\alpha(\theta_2)},
\end{aligned} \tag{4.8}$$

where $\Upsilon_{\theta_2} = \begin{bmatrix} \frac{1}{m_{11}} & -\frac{m_{12}(\theta_2)}{m_{11}\alpha(\theta_2)} \\ 0 & \frac{1}{\alpha(\theta_2)} \end{bmatrix}$ and $(H\Upsilon_{\theta_2})_{ij}$ denotes the component ij of matrix $H\Upsilon_{\theta_2}$, $(i,j=1,2)$.

Observer structure. The proposed observer has the following form

$$\begin{aligned}
\dot{\hat{\theta}}_1 &= \frac{\hat{w}_1}{m_{11}} - \ell k_{o1}\phi_{11}(e_{\theta_1}),\\
\dot{\hat{w}}_1 &= -(H\Upsilon_{\theta_2})_{11}\hat{w}_1 - (H\Upsilon_{\theta_2})_{12}\hat{w}_2 + v_1 - \ell^2 k_{o2}\phi_{12}(e_{\theta_1})+\\
&\qquad - \psi_1(q,\Upsilon_{\theta_2}(\hat{w} + k_{ol}\phi_1(e_\theta))),\\
\dot{\hat{\theta}}_2 &= \frac{\hat{w}_2}{\alpha(\theta_2)} - \frac{\ell l_{o1}\phi_{21}(e_{\theta_2})}{\alpha(\theta_2)},\\
\dot{\hat{w}}_2 &= \frac{m_{11}[-(H\Upsilon_{\theta_2})_{21}\hat{w}_1 - (H\Upsilon_{\theta_2})_{22}\hat{w}_2]}{m_{11}\alpha(\theta_2)}+\\
&\quad - \frac{m_{12}(\theta_2)[-(H\Upsilon_{\theta_2})_{11}\hat{w}_1 - (H\Upsilon_{\theta_2})_{12}\hat{w}_2]}{m_{11}\alpha(\theta_2)}\\
&\quad + \frac{m_{11}(-\psi_2(q,\Upsilon_{\theta_2}(\hat{w} + k_{ol}\phi_1(e_\theta))))}{m_{11}\alpha(\theta_2)}\\
&\quad - \frac{m_{12}(\theta_2)(-\psi_1(q,\Upsilon_{\theta_2}(\hat{w} + k_{ol}\phi_1(e_\theta))))}{m_{11}\alpha(\theta_2)}\\
&\quad + \frac{m_{11}v_2 - m_{12}(\theta_2)v_1}{m_{11}\alpha(\theta_2)} - \frac{\ell^2 l_{o2}\phi_{22}(e_{\theta_2})}{\alpha(\theta_2)},
\end{aligned} \tag{4.9a}$$

where $e_{\theta_1} = \hat{\theta}_1 - \theta_1$, $e_{\theta_2} = \hat{\theta}_2 - \theta_2$, the nonlinearities are

$$\begin{aligned}
\phi_{i1}(e_{\theta_i}) &:= \mu_{i1}\lceil e_{\theta_i}\rfloor^{1/2} + \mu_{i2}\lceil e_{\theta_i}\rfloor,\\
\phi_{i2}(e_{\theta_i}) &:= \frac{\mu_{i1}^2}{2}\lceil e_{\theta_i}\rfloor^0 + \frac{3\mu_{i1}\mu_{i2}}{2}\lceil e_{\theta_i}\rfloor^{1/2} + \mu_{i2}^2\lceil e_{\theta_i}\rfloor, \quad \text{for } i=1,2,
\end{aligned} \tag{4.9b}$$

and the estimated states in original coordinates for (4.5) are given by

$$\begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \end{bmatrix} = \Upsilon_{\theta_2}\begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{m_{11}} & -\frac{m_{12}(\theta_2)}{m_{11}\alpha(\theta_2)} \\ 0 & \frac{1}{\alpha(\theta_2)} \end{bmatrix}\begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \end{bmatrix}. \tag{4.9c}$$

The observer (4.9a) is a copy of the transformed system with nonlinear injection terms ϕ_{i1} and ϕ_{i2}. These injections appear additively in the system and also within the nonlinearity ψ. The discontinuous term $\lceil e_{\theta_i} \rfloor^0 = sign(e_{\theta_i})$ ensures robustness of the observer against bounded UP and the other terms in the nonlinearities ensure finite-time convergence to the real states.

4.4 Results

The gains to design of the proposed observer are

$$k_{o1}, k_{o2}, l_{o1}, l_{o2}, \mu_{11}, \mu_{12}, \mu_{21}, \mu_{22} > 0; \ell \geq 1; \qquad k_{o3}, l_{o3} \in \mathbb{R}. \tag{4.9d}$$

These gains can be obtained from the matrix inequalities (4.10a), (4.10b) given in the following Lemma.

Lemma 3. *For any positive constants b, d_{11}, $\underline{h_{A1}}$, $\underline{h_{B1}}$, $\underline{\alpha(\theta_2)} > 0$, and non negative constants $\overline{(H\Upsilon_{\theta_2})_{11}}$, $\overline{(H\Upsilon_{\theta_2})_{22}}$, $\overline{d_{12}(\theta)\,(H\Upsilon_{\theta_2})_{12}}$, $\Gamma_j \geq 0$, for $j = 1, ..., 4$, there exist constants $\ell \geq 1$; k_{oi}, μ_{mi}, l_{oi}, $\gamma_j > 0$, and positive definite symmetric matrices $P_i > 0$, with $i = 1, 2$, such that*

$$A_2^T P_1 + P_1 A_2 < 0, \quad B_2^T P_2 + P_2 B_2 < 0, \tag{4.10a}$$

$$\begin{bmatrix} \mathit{\Pi}_{P1} & \star & \star & \star & \star & \star & \star & \star \\ 0 & \mathit{\Pi}_{P2} & \star & \star & \star & \star & \star & \star \\ B^T P_1 & 0 & -\gamma_1 \mu_{m2}^2 & \star & \star & \star & \star & \star \\ 0 & B^T P_2 & 0 & -\gamma_2 \mu_{m2}^2 & \star & \star & \star & \star \\ B^T P_1 & 0 & 0 & 0 & -\gamma_3 \mu_{m1}^4 \ell^2 & \star & \star & \star \\ 0 & B^T P_2 & 0 & 0 & 0 & -\gamma_4 \mu_{m1}^4 \ell^2 & \star & \star \\ B^T P_1 & 0 & 0 & 0 & 0 & 0 & -\gamma_5 \mu_{m2}^2 & \star \\ 0 & B^T P_2 & 0 & 0 & 0 & 0 & 0 & -\gamma_6 \mu_{m2}^2 \end{bmatrix} \leq 0, \tag{4.10b}$$

are satisfied, where

$$A_1 = \begin{bmatrix} -k_{o1} & \frac{1}{m_{11}} \\ -k_{o2} & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} -k_{o1} & \frac{1}{m_{11}} \\ -k_{o2} & -\left(\frac{\overline{(H\Upsilon_{\theta_2})_{11}}}{\ell \mu_{12}} + b_1 \right) \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; \tilde{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} -l_{o1} & 1 \\ -l_{o2} & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -l_{o1} & 1 \\ -l_{o2} & -\left(\frac{m_{11}\overline{(H\Upsilon_{\theta_2})_{22}} + \overline{m_{12}(\theta)(H\Upsilon)_{12}}}{\ell m_{11} \mu_{22}} + b_1 \right) \end{bmatrix},$$

$$\begin{aligned} \mathit{\Pi}_{P1} =& \ell \underline{h_{A1}} \left(A_1^T P_1 + P_1 A_1 \right) + \gamma_2 \Gamma_2 B B^T + \gamma_3 \Gamma_3 \tilde{B}^T \tilde{B} + \\ & + \gamma_4 \Gamma_4 \tilde{B}^T \tilde{B} + \frac{\gamma_5 \Gamma_5 + \gamma_6 \Gamma_6}{\ell^4} \Delta_\ell E_k^T E_k \Delta_\ell + \epsilon I, \\ \mathit{\Pi}_{P2} =& \frac{\ell \underline{h_{B1}}}{\underline{\alpha(\theta_2)}} \left(B_1^T P_2 + P_2 B_1 \right) + \gamma_1 \Gamma_1 B B^T + \gamma_3 \Gamma_3 \tilde{B}^T \tilde{B} + \\ & + \gamma_4 \Gamma_4 \tilde{B}^T \tilde{B} + \frac{\gamma_5 \Gamma_5 + \gamma_6 \Gamma_6}{\ell^4} \Delta_\ell E_k^T E_k \Delta_\ell + \epsilon I. \end{aligned}$$

▲

The proof of Lemma 3 is in Appendix C.1

The following theorem is the main result of this Chapter.

Theorem 5. *Suppose that parameters (4.9d) satisfy the feasible matrix inequalities (4.10a), (4.10b), then the estimation states (4.9c) converge in finite time to the velocity $\dot{x}$ in (4.5).* ▲

The proof of Theorem 5 is in Appendix C.2.

4.5 Example

The proposed observer (4.9a) is applied by simulations to the Cart-pendulum system. The nonlinear mathematical model of the cart-pendulum is given by

$$\begin{aligned}
\dot{x}_1 &= x_3, \\
\dot{x}_2 &= x_4, \\
\dot{x}_3 &= \frac{a_1\left(k_1 u - x_4^2 \sin x_2 - k_2 x_3\right) + \left(g \sin x_2 - k_3 x_4\right)\cos x_2}{a_1 a_2 - \cos^2 x_2}, \\
\dot{x}_4 &= \frac{\left(k_1 u - x_4^2 \sin x_2 - k_2 x_3\right)\cos x_2 + a_2\left(g \sin x_2 - k_3 x_4\right)}{a_1 a_2 - \cos^2 x_2}.
\end{aligned} \tag{4.11}$$

where $x_1 \equiv$ cart position $[m]$ and $x_2 \equiv$ pendulum angular position $[rad]$ are measured states, $x_3 \equiv$ cart velocity $\left[\frac{m}{s}\right]$ and $x_4 \equiv$ pendulum angular velocity $\left[\frac{rad}{s}\right]$ are unmeasured states. Moreover, u is the control input $[N]$. The cart-pendulum parameters are obtained from INTECO, 2008 as $m = 0.872\,[Kg]$, $l = 0.011\,[m]$, $f_c = 1\,\left[N\cdot\frac{s}{m}\right]$, $f_p = 1.4\cdot 10^{-4}\,\left[\frac{N\cdot m\cdot s}{rad}\right]$, $J_p = 0.0034\,[Kg\cdot m^2]$, $g = 9.81\,[m/s^2]$, $p_1 = 9.4\,[N]$, $p_2 = -0.548\,[N\cdot s/m]$. Obtaining $a_1 = \frac{J_P}{ml} = 0.3545$, $a_2 = \frac{1}{l} = 90.9091$, $k_1 = \frac{p_1}{ml} = 979.9833$, $k_2 = \frac{f_c - p_2}{ml} = 161.3845$, $k_3 = \frac{f_p}{ml} = 0.0146$.

Writing the system (4.11) in the form (4.1), one obtains $C(x) = \begin{bmatrix} 0 & x_4 \sin x_2 \\ 0 & 0 \end{bmatrix}$, $M(x_2) = \begin{bmatrix} a_2 & -\cos x_2 \\ -\cos x_2 & a_1 \end{bmatrix}$, $G(x) = \begin{bmatrix} 0 \\ -g\sin x_2 \end{bmatrix}$, $H = \begin{bmatrix} k_2 & 0 \\ 0 & k_3 \end{bmatrix}$, $D = \begin{bmatrix} k_1 \\ 0 \end{bmatrix}$, $\psi(x) = 0$, where Assumption P-1 is satisfied. The eigenvalues depending on x_2 of $M(x_2)$ is given by $\frac{a_1 + a_2 \pm \sqrt{(a_1-a_2)^2 + 4\cos(x_2)}}{2}$, from which can obtain $\alpha_1 = \frac{a_1 + a_2 - \sqrt{(a_1-a_2)^2+4}}{2}$, $\alpha_2 = \frac{a_1 + a_2 + \sqrt{(a_1-a_2)^2+4}}{2}$ that satisfy Assumption P-2.

In this case $\Upsilon_{x_2} = \begin{bmatrix} \frac{1}{a_2} & \frac{\cos(x_2)}{a_2\alpha(x_2)} \\ 0 & \frac{1}{\alpha(x_2)} \end{bmatrix}$ and $\alpha(x_2) = \sqrt{\frac{a_1 a_2 - \cos^2(x_2)}{a_2}}$, where $(H\Upsilon_{x_2})_{11} = \frac{k_2}{a_2} \geq 0$, $m_{11}(H\Upsilon_{x_2})_{22} - m_{12}(x_2)(H\Upsilon_{x_2})_{12} = \frac{a_2 k_3}{\alpha(x_2)} + \frac{\cos^2(x_2) k_2}{a_2 \alpha(x_2)} \geq 0$ satisfy Assumption P-3. For Assumption P-4 we use $L_{\tilde{\delta}} = 1.5$. For this system $\psi(x) = 0$, then any matrix $Q < 0$ and $S = R = 0$ satisfy assumption P-5. The following parameters are obtained from the transformation

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} x_1 - \frac{\sin(x_2)}{a_2} \\ x_2 \\ a_2 x_3 - \cos(x_2) x_4 \\ \alpha(x_2) x_4 \end{bmatrix}. \tag{4.12}$$

Solving the matrix inequalities (4.10) with the parameters: $\underline{\alpha(\theta_2)} = \sqrt{\frac{a_1 a_2 - 1}{a_2}} = 0.5861$, $\overline{(H\Upsilon_{\theta_2})_{12}} = \frac{k_2}{a_2}\sqrt{\frac{a_2}{a_1 a_2 - 1}}$, $\overline{d_{12}(\theta_2)} = 1$, $\overline{(H\Upsilon_{\theta_2})_{22}} = k_3\sqrt{\frac{a_2}{a_1 a_2 - 1}}$,

$h_{B1} = \frac{a_2 b_2 \ell \mu_{22}}{a_2 k_3 \sqrt{\frac{a_2}{a_1 a_2 - 1}} + \frac{k_2}{a_2}\sqrt{\frac{a_2}{a_1 a_2 -1}} + a_2 b_2 \ell \mu_{22}}$, $h_{A1} = \frac{b_1}{\frac{(H\Upsilon_{\theta_2})_{11}}{\ell \mu_{12}} + b_1}$, $\Gamma_1 = \left(\frac{k_2}{a_2}\sqrt{\frac{a_2}{a_1 a_2 - 1}}\right)^2 = 9.1755$, $\Gamma_2 = \left(\frac{k_2}{a_2^2}\sqrt{\frac{a_2}{a_1 a_2 - 1}}\right)^2 = 0.0011$, $\Gamma_3 = 4L_{\delta_1}^2$, $\Gamma_4 = 4\left(\frac{a_2 L_{\delta_2} + L_{\delta_1}}{a_2}\sqrt{\frac{a_2}{a_1 a_2 - 1}}\right)^2$.

For $b_1 = 2$, $b_2 = 200$, one obtains $P_1 = \begin{bmatrix} 76.7062 & -8.8802 \\ -8.8802 & 1.7438 \end{bmatrix}$; $P_2 = \begin{bmatrix} 45.5100 & -11.7927 \\ -11.7927 & 18.2595 \end{bmatrix}$; $\ell = 1$; $k_{o1} = 0.9594$; $k_{o2} = 5.1402$; $l_{o1} = 1.1713$; $l_{o2} = 3.0827$; $\mu_{m1} = 3$, $\mu_{m2} = 10$; $\mu_{11} = 3$; $\mu_{12} = 300$; $\mu_{21} = 3$; $\mu_{22} = 10$; $\epsilon = 0.0934$; $\gamma_1 = 0.9297$; $\gamma_2 = 0.8079$; $\gamma_3 = 1.1800$; $\gamma_4 = 0.9828$.

Figure 4.1 illustrates the finite-time convergence of the estimate states to the real states. Figure 4.2 illustrates all estimation errors, and the control input applied to the system.

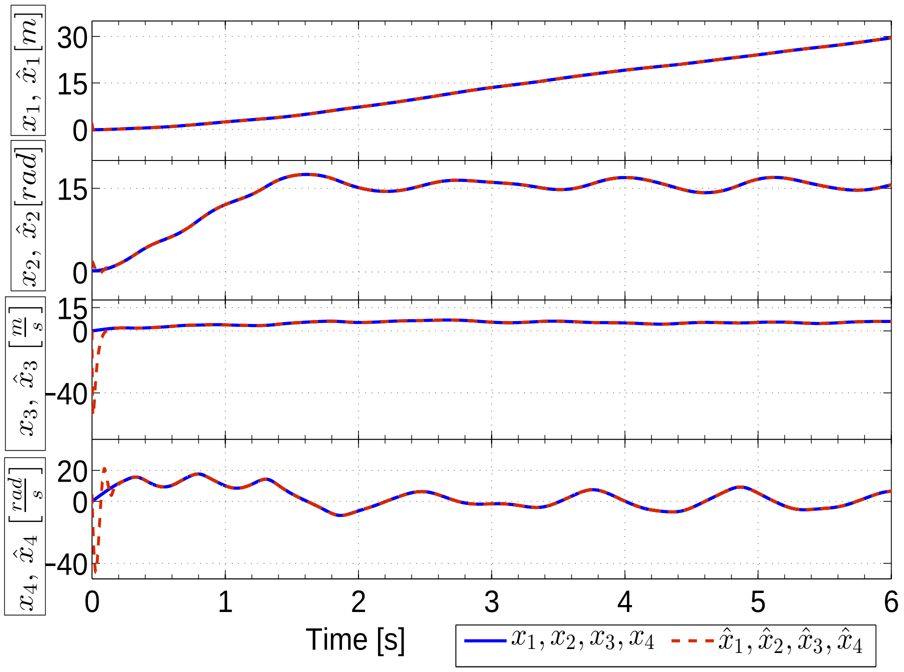

FIGURE 4.1: Real positions x_1 and x_2, estimated positions $\hat{x}_1$ and $\hat{x}_2$, Real velocities x_1 and x_2, estimated velocities $\hat{x}_3$ and $\hat{x}_4$

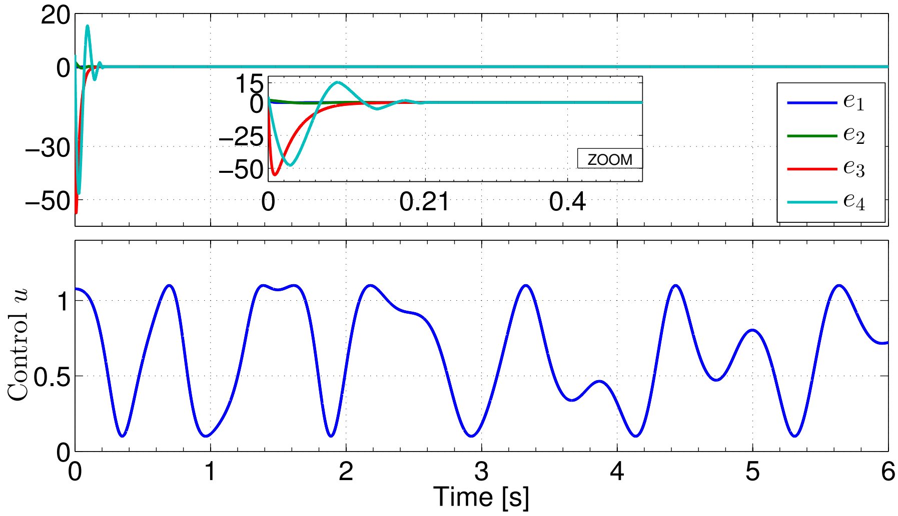

FIGURE 4.2: Estimation errors and the control input

4.6 Conclusions

In this chapter, the dissipative based sliding-mode observer with theoretically exact global and finite-time convergence to the real values of velocities for a class of 2-DOF mechanical systems was proposed. This class of systems may not have the BIBS property and contains Coriolis and centrifugals forces, dry and viscous frictions, and uncertainties/perturbations. The gains of the proposed observer are obtained from the feasible matrix inequalities (4.10). A simulation validation of the proposed observer on a cart-pendulum is presented.

Chapter 5

Experimental implementation: Pendulum-cart system

One of the global observers presented in this work will be used in a mechanical system with the aim of showing its applicability in the experimental framework. Many mechanical systems in the experimental field present several terms not considered in the nominal model such as exogenous perturbations, dry friction, and viscous friction. The last one often is considered as a small and despicable value, but there are mechanical systems where these values are not despised such as the car-pendulum system. A car-pendulum system manufactured by the company INTECO is available in the Sliding-Modes laboratory of the Engineering graduate school at the National Autonomous University of Mexico (UNAM), see Figure 5.1. This car-pendulum is a mechanical system of two degrees of freedom for which the Coriolis force and viscous friction are considered in the model. It is possible to introduce exogenous perturbations through the pendulum position to show the observer robustness along to an discontinuous integral control (Zamora, Moreno, and Kamal, 2013).

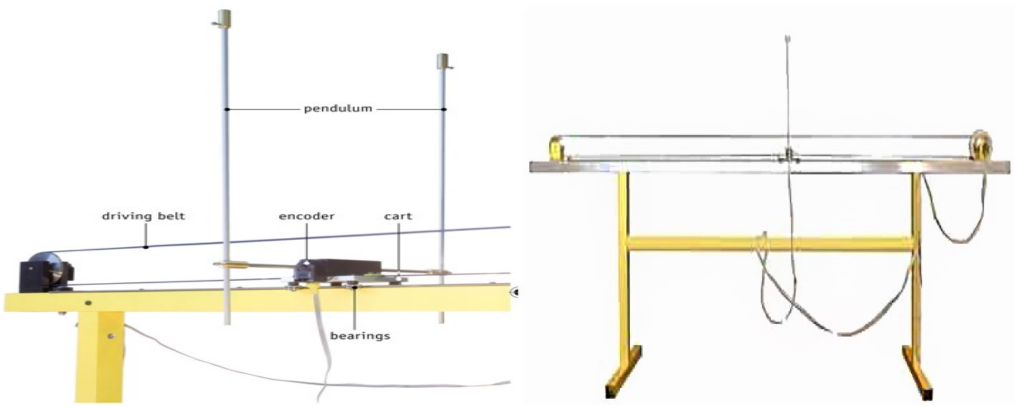

FIGURE 5.1: Sliding-mode laboratory, Faculty of electrical engineering, Universidad Nacional Autónoma de México UNAM

5.1 Model description

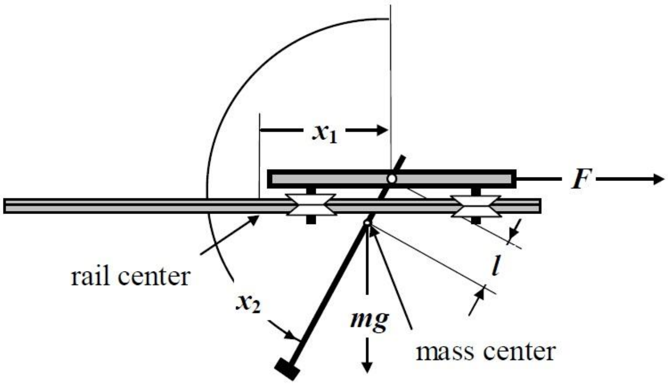

FIGURE 5.2: Pendulum on a cart system

Figure 5.2 shows the functional principle of the system, where the states represent $x_1 \equiv$ cart position $[m]$ and $x_2 \equiv$ pendulum angular position $[rad]$ which are measured, m is the equivalent mass of cart and pendulum, g is the gravity acceleration. The nonlinear mathematical model of the cart-pendulum considering the diagram depicted in Figure 5.2, is given by INTECO, 2008

$$
\begin{aligned}
\dot{x}_1 &= x_3, \\
\dot{x}_2 &= x_4, \\
\dot{x}_3 &= \frac{a_1\left(k_1 u - x_4^2 \sin x_2 - k_2 x_3\right) + \left(g \sin x_2 - k_3 x_4\right)\cos x_2}{a_1 a_2 - \cos^2 x_2}, \\
\dot{x}_4 &= \frac{\left(k_1 u - x_4^2 \sin x_2 - k_2 x_3\right)\cos x_2 + a_2\left(g \sin x_2 - k_3 x_4\right)}{a_1 a_2 - \cos^2 x_2}.
\end{aligned}
\tag{5.1}
$$

where $x_3 \equiv$ cart velocity $\left[\frac{m}{s}\right]$ and $x_4 \equiv$ pendulum angular velocity $\left[\frac{rad}{s}\right]$ which are unmeasured variables. Moreover, u is the control input $[N]$. In Table 5.1, the cart-pendulum parameters given by the manufacturer INTECO, 2008 are shown

	Description	Value
m	Equivalent mass of cart and pendulum	$0.872\,[Kg]$
l	Distance from axis of rotation to center of mass of system	$0.011\,[m]$
f_c	Dynamic cart friction coefficient	$1\,\left[N \cdot \frac{s}{m}\right]$
f_p	Rotational friction coefficient	$1.4 \cdot 10^{-4}\,\left[\frac{N \cdot m \cdot s}{rad}\right]$
J_p	Pendulum inertial moment with respect to rotation axis	$0.0034\,[Kg \cdot m^2]$
g	Gravity acceleration	$9.81\,[m/s^2]$
p_1	Control force to PWM signal ratio	$9.4\,[N]$
p_2	Control force to cart velocity ratio	$-0.548\,[N \cdot s/m]$

TABLE 5.1: Table of original system parameters.

obtaining from them the following terms $a_1 = \frac{J_P}{ml} = 0.3545$, $a_2 = \frac{1}{l} = 90.9091$, $k_1 = \frac{p_1}{ml} = 979.9833$, $k_2 = \frac{f_c - p_2}{ml} = 161.3845$, $k_3 = \frac{f_p}{ml} = 0.0146$.

5.2 Selection of observer parameters

Writing the system (5.1) in the form (4.1), one obtains $M(x_2) = \begin{bmatrix} a_2 & -\cos x_2 \\ -\cos x_2 & a_1 \end{bmatrix}$, $C(x) = \begin{bmatrix} 0 & x_4 \sin x_2 \\ 0 & 0 \end{bmatrix}$, $G(x) = \begin{bmatrix} 0 \\ -g \sin x_2 \end{bmatrix}$, $H = \begin{bmatrix} k_2 & 0 \\ 0 & k_3 \end{bmatrix}$, $D = \begin{bmatrix} k_1 \\ 0 \end{bmatrix}$, $\psi(x) = 0$, where Assumption P-1 is satisfied. The eigenvalues depending on x_2 of $M(x_2)$ are given by $\frac{a_1+a_2 \pm \sqrt{(a_1-a_2)^2+4\cos(x_2)}}{2}$, from which it can be obtained $\alpha_1 = \frac{a_1+a_2-\sqrt{(a_1-a_2)^2+4}}{2}$, $\alpha_2 = \frac{a_1+a_2+\sqrt{(a_1-a_2)^2+4}}{2}$ that satisfy Assumption P-2. In this case $\Upsilon_{x_2} = \begin{bmatrix} \frac{1}{a_2} & \frac{\cos(x_2)}{a_2\alpha(x_2)} \\ 0 & \frac{1}{\alpha(x_2)} \end{bmatrix}$ and $\alpha(x_2) = \sqrt{\frac{a_1 a_2 - \cos^2(x_2)}{a_2}}$, where $(H\Upsilon_{x_2})_{11} = \frac{k_2}{a_2} \geq 0$, $m_{11}(H\Upsilon_{x_2})_{22} - m_{12}(x_2)(H\Upsilon_{x_2})_{12} = \frac{a_2 k_3}{\alpha(x_2)} + \frac{\cos^2(x_2)k_2}{a_2\alpha(x_2)} \geq 0$ satisfy Assumption P-3. For Assumption P-4 we use $L_{\tilde{\delta}} = 1.5$. For this system $\psi(x) = 0$, then any matrix $Q < 0$ and $S = R = 0$ satisfy Assumption P-5. The following parameters are obtained from transformation

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} x_1 - \frac{\sin(x_2)}{a_2} \\ x_2 \\ a_2 x_3 - \cos(x_2)x_4 \\ \alpha(x_2)x_4 \end{bmatrix}. \tag{5.2}$$

Solving the matrix inequalities (4.10) with the parameters: $\underline{\alpha(\theta_2)} = \sqrt{\frac{a_1 a_2 - 1}{a_2}} = 0.5861$, $\overline{(H\Upsilon_{\theta_2})_{12}} = \frac{k_2}{a_2}\sqrt{\frac{a_2}{a_1 a_2 - 1}}$, $\overline{d_{12}(\theta_2)} = 1$, $\overline{(H\Upsilon_{\theta_2})_{22}} = k_3\sqrt{\frac{a_2}{a_1 a_2 - 1}}$, $\underline{h_{B1}} = \frac{a_2 b_2 \ell \mu_{22}}{a_2 k_3 \sqrt{\frac{a_2}{a_1 a_2 - 1}} + \frac{k_2}{a_2}\sqrt{\frac{a_2}{a_1 a_2 - 1}} + a_2 b_2 \ell \mu_{22}}$, $\underline{h_{A1}} = \frac{b_1}{\frac{\overline{(H\Upsilon_{\theta_2})_{11}}}{\ell \mu_{12}} + b_1}$, $\Gamma_1 = \left(\frac{k_2}{a_2}\sqrt{\frac{a_2}{a_1 a_2 - 1}}\right)^2 = 9.1755$, $\Gamma_2 = \left(\frac{k_2}{a_2^2}\sqrt{\frac{a_2}{a_1 a_2 - 1}}\right)^2 = 0.0011$, $\Gamma_3 = 4L_{\delta_1}^2$, $\Gamma_4 = 4\left(\frac{a_2 L_{\delta_2} + L_{\delta_1}}{a_2}\sqrt{\frac{a_2}{a_1 a_2 - 1}}\right)^2$.
For $b_1 = 2$, $b_2 = 200$, one obtains $P_1 = \begin{bmatrix} 76.7062 & -8.8802 \\ -8.8802 & 1.7438 \end{bmatrix}$; $P_2 = \begin{bmatrix} 45.5100 & -11.7927 \\ -11.7927 & 18.2595 \end{bmatrix}$; $\ell = 1$; $k_{o1} = 0.9594$; $k_{o2} = 5.1402$; $l_{o1} = 1.1713$; $l_{o2} = 3.0827$; $\mu_{m1} = 3$, $\mu_{m2} = 10$; $\mu_{11} = 3$; $\mu_{12} = 300$; $\mu_{21} = 3$; $\mu_{22} = 10$; $\epsilon = 0.0934$; $\gamma_1 = 0.9297$; $\gamma_2 = 0.8079$; $\gamma_3 = 1.1800$; $\gamma_4 = 0.9828$.

5.3 Experimental results

The experiment is performed in a cart-pendulum plant developed by INTECO, 2008. The control law is programmed in MATLAB with SIMULINK and the RTWT (Real Time Windows Target) MathWorks Toolbox is included. Through a data acquisition card RT-DAC4/PCI the communication between the plant and a computer is achieved. The control signal is computed by PC and sent to the DC flat motor as a PWM signal.

In the experiment, the pendulum is driven near to $x_0 = (0, 0)$ by means of a swing-up algorithm which is taken from the test examples of the INTECO cart-pendulum system (INTECO, 2008). When the pendulum angular position is in a region less than $0.2\,[rad]$, i.e., $\|x_2\| < 0.2\,[rad]$ the control signal switch from the swing up to an Integral-discontinuous Controller (5.3) proposed in Zamora, Moreno, and Kamal, 2013, which stabilize the origin of the system

$$\begin{aligned} u &= -26\lceil s \rfloor^{1/3} - 25\lceil \dot{s} \rfloor^{1/2} + v, \\ \dot{v} &= -5\, sign(s), \\ s &= x_3 + 0.5\tilde{x}_2 + 4\tilde{x}_1, \end{aligned} \tag{5.3}$$

where $\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} = \begin{bmatrix} 0.0028 & -0.0008 & 0 & -0 \\ 0 & 0 & 0.0028 & -0.0008 \\ 0 & 0.0272 & 0 & 0 \\ 0 & 0 & 0 & 0.0272 \end{bmatrix} x.$

The controller needs information of x_3, x_4 and these are obtained through the proposed observer (4.9). Initial conditions are different to zero for all states and the sample time is $1\ [ms]$.

In experimental development, an impulse-like external force at time $t = 17.8s$ is applied this for illustrating the robustness of the proposed observer. This experiment can also be visualized through the following link https://youtu.be/NNQpEaYbZ0c. The experimental results are shown in Figures 5.3 and 5.4. Figure 5.3 illustrates the positions which converge to the origin. Figure 5.4 shows the velocity estimations obtained by the proposed observer.

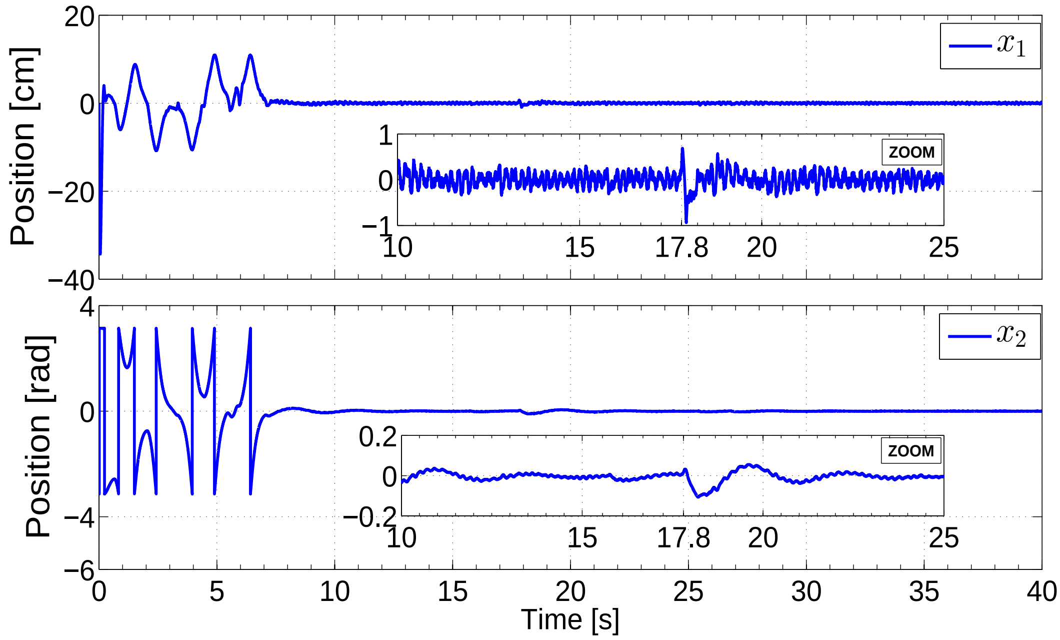

FIGURE 5.3: The cart position and the pendulum angular position

In the cart-pendulum system, the simulation results obtained in Figure 4.1 and Figure 4.2 illustrate the effectiveness of the proposed observer which does not depend of the BIBS property. The experimental results illustrated in Figure 5.3 and Figure 5.4 show the applicability of the observer which is used to the stability of the origin.

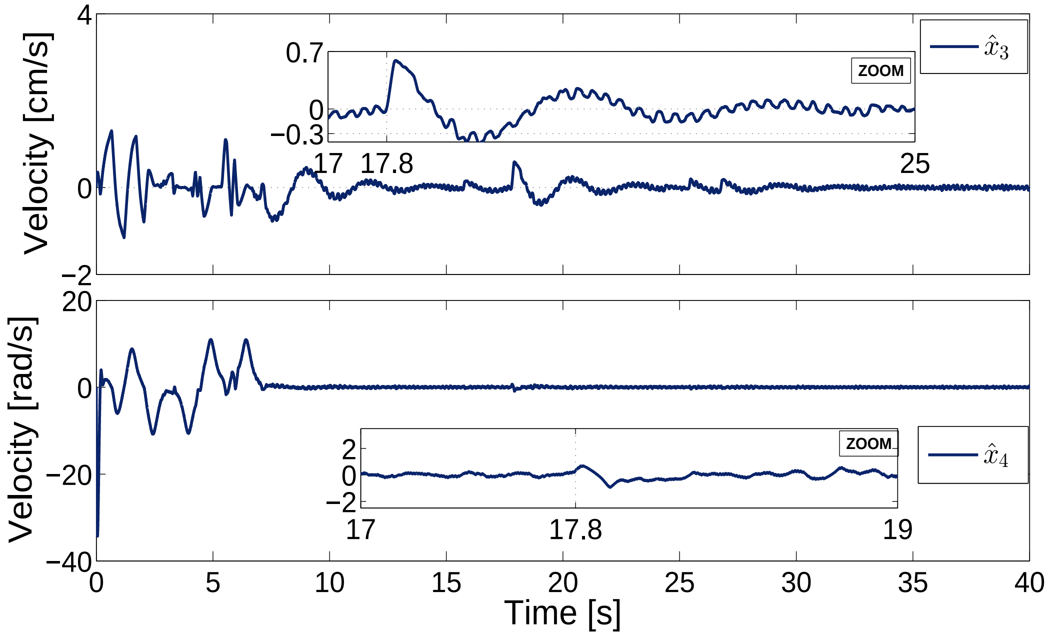

FIGURE 5.4: Estimations of the cart velocity and the pendulum angular velocity obtained by the observer

Chapter 6

Conclusions

The design of observers for uncertain nonlinear systems is currently one of the main topics in control and observation theory. In realistic scenarios, the uncertainty/perturbation (UP) is considered for the design of observers, however this usually does not satisfy the relative-degree-one condition, which is one of the necessary conditions for the existence of observers when the UP is arbitrary. For this reason is necessary to know some characteristics of the UP, which allow the construction of observers for uncertain systems with relative degree greater than one.

In this work, the UP is considered as bounded and two problems in the design of global observers for uncertain nonlinear systems were addressed: i) when the UP has a relative degree greater than one; where the dissipative observers may only ensure convergence to a neighborhood of real states, ii) when the uncertain nonlinear systems do not have the bounded-input-bounded-state property with respect to the UP; where the sliding-mode observers loose their robust properties. These problems were satisfactorily solved for some classes of uncertain nonlinear systems such as a chain of integrators of arbitrary order and a class of mechanical systems, where the theoretically exact convergence in finite time to real states was assured.

Consequently, two structures of observers have been proposed. (a) The first one uses a scaled dissipative stabilizer and a HOSM differentiator under a cascade scheme. This generalizes the structure obtained in the linear case for the nonlinear case, where the order of observer is two times the order of the system. This structure has advantages in the stability analysis because it is obtained from the stability of each substructure. (b) The second one uses the Generalized Super-Twinting algorithm and introduces correction terms in the nonlinearities, where the order of observer is equal to the order of the system. The stability analysis from this structure uses dissipative properties of the nonlinearities through the construction of Lyapunov functions. The second structure allows more flexibility in the design of gains in comparison with the first structure applied to second order systems.

Scaled gains were introduced in the standard structure of the dissipative observer, which we called *scaled dissipative stabilizer* (SDS), see Chapter 2 where a chain of integrators is considered. This SDS allows to obtain the uniformly ultimately bounded property on the estimation error dynamics, where it is explicitly shown that if the observer gains grow, then the ultimate bound is reduced. The first observer structure was applied to a chain of integrators, which combines SDS and a HOSM differentiator under a cascade scheme. The order of the obtained observer is $2n$ for a chain of integrators of order and relative degree n, due to its structure. The independence of the design of the SDS gains and the HOSM differentiator in the cascade scheme was guaranteed.

A topic for future research that follows from these results is to reduce the order of the observer by changing the structure. In this case, another alternative for the stability analysis will be necessary to ensure the finite-time convergence to the real states of the plant.

Towards this end, the second structure was proposed and applied to 1-DOF nonlinear mechanical systems in Chapter 3, and extended to a class of 2-DOF in Chapter 4. Lyapunov-like functions were proposed which consider the dissipative properties of the nonlinearities and ensure finite-time convergence to the velocities. In these classes of nonlinear mechanical systems, Coriolis and centrifugal forces, dry and viscous frictions, perturbations with relative degree two were considered. There are many global observers proposed in the literature which only consider the problem of the presence of Coriolis and centrifugal forces, and those are not applicable for the classes of systems considered in this work. There are finite-time observers proposed in the literature, but they require the BIBS property. Consequently, one of the proposed observers was used in the experimental framework through a car-pendulum system, where its effectiveness, robustness and applicability are shown, see Chapter 5.

A topic for future research that follows from the classes previously mentioned is to extend the results to mechanical systems with n degrees of freedom, where the nonlinearities from the Coriolis forces, viscous and dry frictions and bounded UP are considered.

The global observers proposed in this work have theoretically exact convergence in finite time to the unmeasured states, where gains can be obtained from the matrix inequalities and for which the feasibility was assured.

Appendix A

HOSM observers with SDS for a chain of integrators of arbitrary order: result proofs

A.1 Proof of Lemma 1 (Page 13)

Since the non linearity $\Psi := \Psi(x,h)$ is $\{q,S,R\}-$dissipative, i.e.

$$\begin{bmatrix} \Psi \\ h \end{bmatrix}^T \begin{bmatrix} q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \Psi \\ h \end{bmatrix} = \Psi^T q \Psi + \Psi^T S h + h^T S^T \Psi + h^T R h \geq 0.$$

From which one gets the following relationships:

$$-q\|\Psi\|^2 \leq \Psi^T S h + h^T S^T \Psi + h^T R h \leq 2\|\Psi\|\|h\| \left\{\lambda_{\max}(S^T S)\right\}^{1/2} + \lambda_{\max}(R)\|h\|^2.$$

Consequently, one has

$$\begin{aligned} \|\Psi\| \leq & \left(\left\{ -\frac{\lambda_{\max}(R)}{q} + \frac{\lambda_{\max}(S^T S)}{q^2} \right\}^{1/2} - \frac{\{\lambda_{\max}(S^T S)\}^{1/2}}{q} \right) \|h\|, \\ \leq & \frac{1}{-q} \left(\left\{ -\lambda_{\max}(R) q + \lambda_{\max}(S^T S) \right\}^{1/2} + \{\lambda_{\max}(S^T S)\}^{1/2} \right) \|h\|, \end{aligned}$$

and the inequality

$$\begin{bmatrix} \Psi \\ h \end{bmatrix}^T \begin{bmatrix} -q^2 & 0 \\ 0 & \tilde{R} \end{bmatrix} \begin{bmatrix} \Psi \\ h \end{bmatrix} \geq 0,$$

is satisfied where

$$\tilde{R} = \left(\left\{ -\lambda_{\max}(R) q + \lambda_{\max}(S^T S) \right\}^{1/2} + \{\lambda_{\max}(S^T S)\}^{1/2} \right)^2 I_{n-1}.$$

□

A.2 Proof of Theorem 1 (Page 14)

Consider $e_v = (e_1, e_2)$, where $e_1 = v_1 - x_1$, $e_2 = v_2 - x_2$, which are obtained from (2.1) with $n = 2$ and (2.8). One obtains

$$\begin{aligned} \dot{e}_1 &= e_2 + l_1 e_1, \\ \dot{e}_2 &= \Psi + l_2 e_1 - w, \end{aligned} \tag{A.1}$$

where $\Psi = \psi(x_1, x_2 + e_2 + l_3 e_1) - \psi(x_1, x_2)$.

The proof of Theorem 1 consists of following two lemmas.

Lemma 4. *For systems (A.1), the positive definite function $V(e) = e^T P e$, fulfills*

$$\dot{V}(e_v) < 0, \quad for \quad \|e_v\| > \frac{2\varrho_w \|(p_2, p_3)\|}{\epsilon}. \tag{A.2}$$

Proof of Lemma 4. Let $V(e_v) = e_v^T P e_v$ be a positive definite function, where $e_v = \begin{bmatrix} e_1 & e_2 \end{bmatrix}^T$. Deriving along the trajectories of system (A.1),

$$\begin{aligned}
\dot{V}(e_v) &= \begin{bmatrix} e_1 \\ e_2 \\ \Psi \end{bmatrix}^T \begin{bmatrix} 2(l_1 p_1 + l_2 p_2) & \star & \star \\ p_1 + l_1 p_2 + l_2 p_3 & 2p_2 & \star \\ p_2 & p_3 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \Psi \end{bmatrix} - 2w(p_2 e_1 + p_3 e_2), \\
&\leq \begin{bmatrix} e_1 \\ e_2 \\ \Psi \end{bmatrix}^T \begin{bmatrix} -r l_3^2 - \epsilon & \star & \star \\ -r l_3 & -r - \epsilon & \star \\ s l_3 & s & -q \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \Psi \end{bmatrix} + 2|w| \cdot \|(p_2, p_3)\| \cdot \|(e_1, e_2)\|, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad p_m = \|(p_2, p_3)\| \\
&\leq - \begin{bmatrix} \Psi \\ e_2 + l_3 e_1 \end{bmatrix}^T \begin{bmatrix} q & s \\ s & r \end{bmatrix} \begin{bmatrix} \Psi \\ e_2 + l_3 e_1 \end{bmatrix} - \epsilon e_v^T e_v + 2\varrho_w p_m \|e_v\|, \\
&\leq - \left(\epsilon \|e_v\| - 2\varrho_w p_m\right) \|e_v\|.
\end{aligned}$$

For $\|e_v\| > \dfrac{2\varrho_w p_m}{\epsilon}$, one has $\dot{V}(e_v) < 0$. □

Lemma 5. *The trajectories in (A.1) converge in finite time T_0 to the compact region $V^{-1}([0, \lambda_{\max}(P)\mu^2])$ contained in $D = \left\{ z \in \mathbb{R}^2 : \|z\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \mu \right\}$, where $\mu = \frac{2\varrho_w p_m + \delta}{\epsilon}$ and $\delta > 0$ is an arbitrary constant. An upper bound T for T_0, is given by:*

- $T = 0$, *if* $e_v(t_0) \in V^{-1}([0, \lambda_{\max}(P)\mu^2])$;
- $T = \dfrac{\lambda_{\min}(P)c^2 - \lambda_{\max}(P)\mu^2}{\epsilon\mu^2 - 2\varrho_w p_m \mu} > 0$,
  if $e_v(t_0) \notin V^{-1}([0, \lambda_{\max}(P)\mu^2])$ *and* $\|e_v(t_0)\| \leq \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}} c$, *for some sufficiently large constant* c.

Proof of Lemma 5. The set $V^{-1}([0, \lambda_{\max}(P)\mu^2])$ is a positively invariant set, because for all $e_v \in fr\left(V^{-1}([0, \lambda_{\max}(P)\mu^2])\right)$ it is satisfied that $\dot{V}(e(t)) < 0$. The function $f(\tau) = \epsilon\tau^2 - 2\varrho_w p_m$ defined in $\tau > \frac{2\varrho_w p_m}{\epsilon} > 0$ is positive and satisfies $f'(\tau) > 0$. Thus, $\alpha = \epsilon\mu^2 - 2\varrho_w p_m \mu = \min_{\mu \leq \|e_v\| \leq c} \left\{ \epsilon \|e\|^2 - 2\varrho_w p_m \|e_v\| \right\}$.

For all $t \geq t_0$ such that

$$e_v(t) \in V^{-1}([0, \lambda_{\min}(P)c^2]) \backslash V^{-1}([0, \lambda_{\max}(P)\mu^2)) \subset \{e_v \in \mathbb{R}^2 : \mu \leq \|e\| \leq c\},$$

one has

$$\begin{aligned}
&\dot{V}(e_v(t)) < -\left(\epsilon \|e_v\|^2 - 2\varrho_w p_m \|e_v\|\right) < -\alpha, \\
&V(e_v(t)) < V(e_v(t_0)) - \alpha(t - t_0) \leq \lambda_{\min}(P)c^2 - \alpha(t - t_0),
\end{aligned}$$

after $T = \frac{\lambda_{\min}(P)c^2 - \lambda_{\max}(P)\mu^2}{\alpha} > 0$ starting from t_0, the trajectory is in $V^{-1}([0, \lambda_{\max}(P)\mu^2])$. □

From Lemma 4 and Lemma 5, the inequality

$$\begin{aligned}
|\ddot{e}_1| &= |\psi(x_1, x_2 + e_2 + l_3 e_1) - \psi(x_1, x_2) + (l_1^2 + l_2)e_1 + l_1 e_2 - w|, \\
&\leq L_2 |e_2 + l_3 e_1| + |w| + |(l_1^2 + l_2)e_1 + l_1 e_2|, \\
&\leq L_2 R\sqrt{l_3^2 + 1} + \varrho_w + R\sqrt{(l_1^2 + l_2)^2 + l_1^2}, \\
&= R\left(L_2\sqrt{l_3^2 + 1} + \sqrt{(l_1^2 + l_2)^2 + l_1^2}\right) + \varrho_w = L_f,
\end{aligned} \tag{A.3}$$

is valid for $t \geq t_0 + T_0$. A Lipschitz constant of $\ddot{e}_1$ is estimated by L_f. This allows us to use the sliding-mode differentiator (1.1) to determine $\dot{e}_1$, where z_1 and z_2 converge in finite time to e_1 and $\dot{e}_1$, respectively.

Consequently,

$$\tilde{x}_1 := v_1 - z_0 \quad \text{and} \quad \tilde{x}_2 := v_2 + l_1 e_1 - z_1,$$

converge to the states x_1 and x_2 in finite time, respectively. □

A.3 Proof of Theorem 2 (Page 16)

i) Consider a particular choice of parameters as follows: Fix the parameters $Q = Q^T > 0$, $\tilde{N}$ and $\epsilon > 0$, and choose K such that A_K is a Hurwitz matrix. Find a matrix $P = P^T > 0$ such that the Lyapunov inequality $PA_K + A_K^T P + \epsilon Q + (\tilde{I}_n + \tilde{N}C)^T R(\tilde{I}_n + \tilde{N}C) < 0$, is satisfied. As the parameter l appears only in the main diagonal of the matrix inequality (2.15), there exists a parameter $l_0 \geq 1$ such that the matrix inequality (2.15) is satisfied for all $l \geq l_0$.

ii) Consider the transformation

$$\zeta := l^n \Delta_l^{-1} e_v, \tag{A.4}$$

where its time derivative is given by

$$\begin{aligned}
\dot{\zeta} &= l^n \Delta_l^{-1}(A + \Delta_l KC) l^{-n} \Delta_l \zeta + l^n \Delta_l^{-1} B\left(\psi(v + NCe_v) - \psi(x) - w\right), \\
&= l\left(A + KC\right)\zeta + \frac{l^n}{l^n} B\left(\Psi(x, (\tilde{I}_n + \tilde{N}C)e_v) - w\right), \\
&= l\left(A + KC\right)\zeta + B\left(\Psi(x, l^{-n}(\tilde{I}_n + \tilde{N}C)\Delta_l \zeta) - w\right).
\end{aligned} \tag{A.5}$$

Let $V_P(\zeta) := \zeta^t P\zeta$ and $V_Q(\zeta) := \zeta^T Q\zeta$, where the derivative of V_P along trajectories of (A.5) is given as

$$\begin{aligned}\dot{V}_P(\zeta) &= l\zeta^T\left[(A+KC)^T P + P(A+KC)\right]\zeta + \Psi B^T P\zeta + \zeta^T PB\Psi - 2\zeta^T PBw,\\ &= \begin{bmatrix}\zeta\\ \Psi\end{bmatrix}^T \begin{bmatrix} l\left[(A+KC)^T P + P(A+KC)\right] & PB\\ B^T P & 0\end{bmatrix}\begin{bmatrix}\zeta\\ \Psi\end{bmatrix} - 2\zeta^T PBw,\\ &\le \begin{bmatrix}\zeta\\ \Psi\end{bmatrix}^T \begin{bmatrix} l\left[A_K^T P + PA_K\right] + \frac{l}{l^{2n}}(\tilde{I}_n\Delta_l)^T_{l\tilde{N}} R(\tilde{I}_n\Delta_l)_{l\tilde{N}} & \star\\ B^T P + \frac{l}{l^{2n}} S(\tilde{I}_n\Delta_l)_{l\tilde{N}} & lq\end{bmatrix}\begin{bmatrix}\zeta\\ \Psi\end{bmatrix} - 2\zeta^T PBw,\\ &\le -l\epsilon V_Q(\zeta) + 2\frac{V_Q^{1/2}(\zeta)}{\sqrt{\lambda_{\min}(Q)}}\|PB\||w|,\\ &\le -V_Q^{1/2}(\zeta)\left(l\epsilon V_Q^{1/2}(\zeta) - \frac{2\|PB\|\varrho_w}{\sqrt{\lambda_{\min}(Q)}}\right).\end{aligned}$$

Let $\mu := \frac{\varrho_w(2\|PB\|+\delta)}{l\epsilon\sqrt{\lambda_{\min}(Q)}}$, where $\delta > 0$ is an arbitrary scalar.
The set $V_P^{-1}\left(\left[0, \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\mu^2\right]\right)$ is a positive-invariant set. The trajectories of ζ converge to the compact region $V_P^{-1}\left(\left[0, \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\mu^2\right]\right)$. The function $f(\tau) := l\epsilon\tau^2 - \frac{2\varrho_w\|PB\|}{\sqrt{\lambda_{\min}(Q)}}\tau$ is positive for $\tau \ge \mu$, and it also satisfies $f'(\tau) = 2l\epsilon\tau - \frac{2\varrho_w}{\sqrt{\lambda_{\min}(Q)}}\|PB\| > 0$. Therefore,

$$\beta := \epsilon l\mu^2 - \frac{2\varrho_w\|PB\|}{\sqrt{\lambda_{\min}(Q)}}\mu = \min_{\mu\le V_Q^{1/2}(\zeta)}\left\{l\epsilon V_Q(\zeta) - 2\frac{V_Q^{1/2}(\zeta)}{\sqrt{\lambda_{\min}(Q)}}\|PB\|\varrho_w\right\} > 0. \tag{A.6}$$

For all $t \ge t_0$ such that $V_P(\zeta(t)) \ge \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\mu^2$, one has

$$\begin{aligned}\dot{V}_P(\zeta(t)) &\le -\left(l\epsilon V_Q(\zeta) - 2\frac{V_Q^{1/2}(\zeta)}{\sqrt{\lambda_{\min}(Q)}}\|PB\|\varrho_w\right) \le -\beta,\\ V_P(\zeta(t)) &\le V_P(\zeta(t_0)) - \beta(t - t_0).\end{aligned}$$

After the time $T := \frac{V(\zeta(t_0)) - \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\mu^2}{\beta}$ starting from t_0, the trajectories are in $V_P^{-1}\left(\left[0, \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\mu^2\right]\right)$.
Consider the diagonalization of P

$$\zeta^T P\zeta = \zeta^T G^T DG\zeta = \alpha^T D\alpha, \tag{A.7}$$

where G is an orthogonal matrix and $D = diag\{\lambda_i(P)\}$ with $\lambda_i(P)$ the eigenvalues of the matrix P for $i = 1, ..., n$.

One obtains

$$
\begin{aligned}
e_v =& \frac{1}{l^n}\Delta_l \zeta = \frac{1}{l^n}\Delta_l G^T \alpha = \frac{1}{l^n}\Delta_l G^T \mu \ diag\left\{\sqrt{\frac{\lambda_{\max}(P)}{\lambda_i(P)\lambda_{\min}(Q)}}\right\}\tilde{\alpha},\\
=& \frac{\varrho_w(2\|PB\|+\delta)}{l\epsilon\sqrt{\lambda_{\min}(Q)}}\frac{\Delta_l}{l^n}G^T \ diag\left\{\sqrt{\frac{\lambda_{\max}(P)}{\lambda_i(P)\lambda_{\min}(Q)}}\right\}\tilde{\alpha},
\end{aligned}
$$

where $\|\tilde{\alpha}\| \leq \sqrt{n}$ and $\frac{\Delta_l}{l^n} \leq I_n$. Then, we obtain

$$
\|e_v\| \leq \frac{\varrho_w(2\|PB\|+\delta)}{l\epsilon\lambda_{\min}(Q)}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\,n. \tag{A.8}
$$

□

A.4 Proof of Theorem 3 (Page 18)

The following Lemma ensures that the n-th derivative of the measured output of the dissipative error is bounded and that its bound is independent of the growth of the dissipative observer gains.

Lemma 6. *If A, B, Δ_l, K and C are defined as in (2.2), then*

1) $(A+\Delta_l KC)^m = l^m \Delta_l (A+KC)^m \Delta_l^{-1}$,
2) $C(A+\Delta_l KC)^m = l^{m+1} C(A+KC)^m \Delta_l^{-1} \quad$ *for* $\quad m \in \mathbb{N}$,
3) $C(A+\Delta_l KC)^{n-1} B = 1$.

Proof of Lemma 6. 1) For induction, $m = 1$

$$
\begin{aligned}
(A+\Delta_l LC) =& \Delta_l(\Delta_l^{-1}A + LC),\\
=& \Delta_l(lA\Delta_l^{-1} + LC),\\
=& \Delta_l(lA + LC\Delta_l)\Delta_l^{-1},\\
=& \Delta_l(lA + lLC)\Delta_l^{-1},\\
=& l\Delta_l(A+LC)\Delta_l^{-1}.
\end{aligned}
$$

Assume that this is valid for $m = k$, then

$$
\begin{aligned}
(A+\Delta_l LC)^{k+1} =& (A+\Delta_l LC)^k (A+\Delta_l LC),\\
=& l^k \Delta_l (A+LC)^k \Delta_l^{-1} l \Delta_l (A+LC)\Delta_l^{-1},\\
=& l^{k+1}\Delta_l (A+LC)^{k+1}\Delta_l^{-1}.
\end{aligned}
$$

2)

$$
C(A+\Delta_l LC)^m = l^m C\Delta_l (A+LC)^m \Delta_l^{-1} = l^{m+1} C(A+LC)^m \Delta_l^{-1}.
$$

3) This happens because the matrix A is a shift function, i.e. for a vector $\eta := [\eta_1, \eta_2, ..., \eta_n]^T$ we obtain

$$
A\eta = [\eta_2, ..., \eta_n, 0]^T \quad \text{and} \quad \eta^T A = [0, \eta_1, ..., \eta_{n-1}], \tag{A.9}
$$

$$(A+\Delta_l LC)^r = \begin{bmatrix} * & \cdots & ll_1 & 1 & 0 & \cdots \\ * & \cdots & l^2 l_2 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ * & \cdots & l^n l_n & 0 & 0 & \vdots \end{bmatrix}, \quad for \ 1 \le r \le n, \tag{A.10}$$

Consequently, $C(A+\Delta_l LC)^{n-1}B = 1$.

□

Proof of Theorem 3. Consider the diagonalization of P

$$\zeta^T P \zeta = \zeta^T G^T D G \zeta = \alpha^T D \alpha, \tag{A.11}$$

where G is an orthogonal matrix and $D = diag\{\lambda_i(P)\}$ and $\lambda_i(P)$ represents the eigenvalues of the matrix P for $i = 1, ..., n$. Analyzing the Lipschitz condition for the (n-1)-th derivative of Ce_v, one gets

$$\begin{aligned}
e_{vy} &= Ce_v, \\
\dot{e}_{vy} &= C(A+\Delta_l KC)e_v, \\
&\vdots \\
e_{vy}^{(n-1)} &= C(A+\Delta_l KC)^{n-1}e_v, \\
e_{vy}^{(n)} &= C(A+\Delta_l KC)^n e_v + C(A+\Delta_l KC)^{n-1}B\,\Psi - C(A+\Delta_l KC)^{n-1}B\,w, \\
&= l^{n+1}(A+KC)^n \Delta_l^{-1} l^{-n} \Delta_l \zeta + \Psi - w, \\
&= l(A+KC)^n G^T \mu \ diag\left\{\sqrt{\frac{\lambda_{\max}(P)}{\lambda_i(P)\lambda_{\min}(Q)}}\right\}\tilde{\alpha} + \Psi - w, \\
&= \frac{\varrho_w(2\|PB\|+\delta)}{\epsilon\lambda_{\min}(Q)}(A+KC)^n G^T \ diag\left\{\sqrt{\frac{\lambda_{\max}(P)}{\lambda_i(P)}}\right\}\tilde{\alpha} + \Psi - w,
\end{aligned}$$

where $\alpha := \mu \ diag\left\{\sqrt{\frac{\lambda_{\max}(P)}{\lambda_i(P)\lambda_{\min}(Q)}}\right\}\tilde{\alpha}$ and $\|\tilde{\alpha}\| \le \sqrt{n}$. This satisfies the conditions for using the high-order sliding-mode differentiator (2.18).

The structure of the HOSM differentiator (2.17b) allows us to verify that $e_{zy} = 0$ is satisfied from $Cz - e_{vy} = 0$. The estimation error of the HOSM observer with SDS satisfies

$$\begin{aligned}
e =& \hat{x} - x, \\
=& v - \mathscr{O}^{-1}z - x, \\
=& e_v - \mathscr{O}^{-1}z, \\
=& e_v - \mathscr{O}^{-1}\left(z - \begin{bmatrix} e_{vy} & \dot{e}_{vy} & \cdots & \dot{e}_{vy}^{(n-1)} \end{bmatrix}^T\right) - \mathscr{O}^{-1}\begin{bmatrix} e_{vy} & \dot{e}_{vy} & \cdots & \dot{e}_{vy}^{(n-1)} \end{bmatrix}^T, \\
=& e_v - \mathscr{O}^{-1}e_{zy} - \mathscr{O}^{-1}\left(\mathscr{O}e_v\right), \\
=& -\mathscr{O}^{-1}e_{zy}.
\end{aligned}$$

□

Appendix B

Dissipative approach to global SM observers design for 1-DOF mechanical systems: result proofs

B.1 Proof of Lemma 2 (Page 31)

Choosing the positive parameters k_1, $k_2 > 0$, it is possible to ensure that the matrix $A = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix}$ is Hurwitz. Thus, solving the following Lyapunov inequality where the parameters θ_2, $\epsilon > 0$; $k_4 < 0$, and k_3 are fixed

$$\underbrace{A^T P + PA + H_E + \theta_2 H_C + \bar{H}_E + \epsilon I}_{\tilde{A}} < 0, \tag{B.1}$$

the parameters p_1 and θ_3 are obtained from P. If μ_2 is chosen such that $\mu_2 > \sqrt{\theta_1} + \theta_3$ is satisfied, then $\tilde{\theta}_1, \tilde{\theta}_3 \leq 1$. By Schur's complement in (3.16) using (B.1) one obtains

$$\begin{bmatrix} \theta_1 p & 0 \\ 0 & -\theta_2 \left(\frac{\mu_1}{2d_2} \right)^2 \end{bmatrix} - \begin{bmatrix} B^T P \\ B^T P \end{bmatrix} \tilde{A}^{-1} \begin{bmatrix} PB & PB \end{bmatrix} \leq 0,$$

which ensures that the inequality (3.16) is satisfied for θ_1, $\mu_1 > 0$ sufficiently large. □

For the proof of Theorem 4, a definition and a previous result are required, which are given as follows.

Definition 2 (Rocha-Cózatl and Moreno, 2011). *A time-varying nonlinearity $\gamma : [0, \infty) \times \mathbb{R}^p \to \mathbb{R}^m$ with $\gamma(t, \upsilon)$, piecewise continuous in t, locally Lipschitz in υ and $\gamma(t, 0) = 0$, is called $\{Q, S, R\}$-dissipative if for each $t \geq 0$ and $\upsilon \in \mathbb{R}^p$ the following inequality is satisfied*

$$\omega(\gamma(t,\upsilon), \upsilon) = \begin{bmatrix} \gamma(t,\upsilon) \\ \upsilon \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \gamma(t,\upsilon) \\ \upsilon \end{bmatrix} \geq 0, \tag{B.2}$$

where $Q = Q^T \in \mathbb{R}^{m \times m}$, $S \in \mathbb{R}^{m \times p}$, $R = R^T \in \mathbb{R}^{p \times p}$. ◇

Lemma 7. *If a non linearity $\gamma(t, \upsilon)$ is $\{Q, S, R\}-$dissipative with Q a negative definite matrix, then there exists a matrix $\tilde{R}$ such that $\gamma(t, \upsilon)$ is $\{Q, 0, \tilde{R}\}-$dissipative.* ▲

Proof of Lemma 7. Since the non linearity $\gamma := \gamma(t, \upsilon)$ is $\{Q, S, R\}-$dissipative where $Q < 0$, it is satisfied that

$$\begin{bmatrix} \gamma \\ \upsilon \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \gamma \\ \upsilon \end{bmatrix} = \gamma^T Q \gamma + \gamma^T S \upsilon + \upsilon^T S^T \gamma + \upsilon^T R \upsilon \geq 0.$$

From which we get the following inequalities:

$$\gamma^T S\upsilon + \upsilon^T S^T\gamma + \upsilon^T R\upsilon \geq -\gamma^T Q\gamma \geq -\lambda_M(Q)\|\gamma\|^2, \tag{B.3}$$

$$2\|\gamma\|\|\upsilon\|\left\{\lambda_M(S^TS)\right\}^{1/2} + \lambda_M(R)\|\upsilon\|^2 \geq -\lambda_M(Q)\|\gamma\|^2,$$

where the matrix property $\lambda_m(Q)\|\gamma\|^2 \leq \gamma^T Q\gamma \leq \lambda_M(Q)\|\gamma\|^2$ is used.

Consider the auxiliary inequality $bxy + cy^2 \geq ax^2$, where $a, b, c > 0$, $x, y \geq 0$. Then from $cy^2 \geq -bxy + ax^2$ by completing squares one gets $\left(c/a + b^2/(4a^2)\right) y^2 \geq x^2 - (b/a)xy + (b/(2a))^2y^2 = (x - (b/2a)y)^2$. Consequently, the inequality

$$\left(\sqrt{c/a + b^2/(4a^2)} + (b/2a)\right) y \geq x,$$

is obtained. Now substituting $x = \|\gamma\|$, $y = \|\upsilon\|$, $a = -\lambda_M(Q)$, $b = 2\left\{\lambda_M(S^TS)\right\}^{1/2}$, $c = \lambda_M(R)$ one gets

$$\left(\sqrt{-\frac{\lambda_M(R)}{\lambda_M(Q)} + \frac{\lambda_M(S^TS)}{\lambda_M^2(Q)}} - \frac{\{\lambda_M(S^TS)\}^{1/2}}{\lambda_M(Q)}\right)\|\upsilon\| \geq \|\gamma\|,$$

$$\frac{\sqrt{-\lambda_M(R)\lambda_M(Q) + \lambda_M(S^TS)} + \{\lambda_M(S^TS)\}^{1/2}}{-\lambda_M(Q)}\|\upsilon\| \geq \|\gamma\|.$$

According to the inequality $\frac{1}{\lambda_m(Q)}\gamma^T Q\gamma \leq \|\gamma\|^2$ we obtain

$$\tilde{R} = \frac{-\lambda_m(Q)}{\lambda_M^2(Q)}\left(\sqrt{-\lambda_M(R)\lambda_M(Q) + \lambda_M(S^TS)} + \sqrt{\lambda_M(S^TS)}\right)^2 I_p,$$

where I_p is the identity matrix of size p. □

B.2 Proof of Theorem 4 (Page 31)

The error dynamics between the systems (4.8) and (4.9a) with $e_1 = \hat{x}_1 - x_1$, $e_2 = \hat{x}_2 - x_2$ is given by

$$\begin{cases} \dot{e}_1 = e_2 - k_1\phi_1(e_1), \\ \dot{e}_2 = \Upsilon(y)\ (\Phi + \Psi - w) - k_2\phi_2(e_1), \end{cases} \tag{B.4}$$

$$\text{where}\ \ \Phi := \varphi\left(y, (\Upsilon(y))^{-1}x_2 + z\right) - \varphi\left(y, (\Upsilon(y))^{-1}x_2\right),$$
$$\Psi := \psi\left(y, (\Upsilon(y))^{-1}x_2 + \bar{z}\right) - \psi\left(y, (\Upsilon(y))^{-1}x_2\right),$$

and $z = (\Upsilon(y))^{-1}(e_2 + k_3\phi_1(e_1))$, $\bar{z} = (\Upsilon(y))^{-1}(e_2 + k_4\phi_1(e_1))$.

Consider the vector $\zeta := \begin{bmatrix} \phi_1(e_1) & e_2 \end{bmatrix}^T$, and a Lyapunov function candidate $V(e) = \zeta^T P\zeta$. Before analyzing the time derivative of $V(e)$, we identify the dissipative properties of the non-linearities $\frac{\Upsilon(y)w}{\phi_1'(e_1)}$, $\frac{\Upsilon(y)}{\phi_1'(e_1)}\Phi$ and $\frac{\Upsilon(y)}{\phi_1'(e_1)}\Psi$ with respect to ζ, where $\phi_1'(e_1) = \frac{\mu_1}{2}|e_1|^{-1/2} + \mu_2$.

$a)$ The inequality $\left(\Upsilon(y)\frac{L_w}{\phi_2(e_1)}\right)^2 \cdot \phi_1^2(e_1) - \left(\frac{-\Upsilon(y)w}{\phi_1'(e_1)}\right)^2 \geq 0$, is satisfied from the condition $|w(t)| \leq L_w$ defined in (3.8). Notice that $\phi_2 = \phi_1'\phi_1$ is satisfied. The function ϕ_2

satisfies the condition $|\phi_2| \geq \frac{\mu_1^2}{2}$, we can obtain the following dissipative property

$$\underbrace{\begin{bmatrix} \frac{-\Upsilon(y)w(t)}{\phi_1'(e_1)} \\ \zeta \end{bmatrix}^T \begin{bmatrix} -\left(\frac{\mu_1^2}{2d_2}\right)^2 & 0 \\ 0 & H_C \end{bmatrix} \begin{bmatrix} \frac{-\Upsilon(y)w(t)}{\phi_1'(e_1)} \\ \zeta \end{bmatrix}}_{\omega\left(\frac{\Upsilon(y)w}{\phi_1'(e_1)},\zeta\right)} \geq 0. \tag{B.5}$$

b) The following dissipative inequality is obtained from Assumption A3-i) applying Lemma 7

$$\begin{bmatrix} \frac{\Upsilon(y)}{\phi_1'(e_1)}\Phi \\ \frac{\Upsilon(y)h}{\phi_1'(e_1)} \end{bmatrix}^T \begin{bmatrix} p & 0 \\ 0 & \left(\frac{|s|+\sqrt{-pr+s^2}}{\sqrt{-p}}\right)^2 \end{bmatrix} \begin{bmatrix} \frac{\Upsilon(y)}{\phi_1'(e_1)}\Phi \\ \frac{\Upsilon(y)h}{\phi_1'(e_1)} \end{bmatrix} \geq 0.$$

When h takes the value of $z = (\Upsilon(y))^{-1}(e_2 + k_3\phi_1(e_1)) = (\Upsilon(y))^{-1}E\zeta$ defined in (B.4), as the function ϕ_1 satisfies $\phi_1'(e_1) \geq \mu_2 > 0$ with $e_1 \neq 0$, the following inequality is satisfied

$$\underbrace{\begin{bmatrix} \frac{\Upsilon(y)}{\phi_1'(e_1)}\Phi \\ \zeta \end{bmatrix}^T \begin{bmatrix} p & 0 \\ 0 & H_E \end{bmatrix} \begin{bmatrix} \frac{\Upsilon(y)}{\phi_1'(e_1)}\Phi \\ \zeta \end{bmatrix}}_{\omega\left(\frac{\Upsilon(y)}{\phi_1'(e_1)}\Phi,\zeta\right)} \geq 0. \tag{B.6}$$

c) The following dissipative inequality is obtained from Assumption A3-ii)

$$\underbrace{\begin{bmatrix} \frac{\Upsilon(y)}{\phi_1'(e_1)}\Psi \\ \zeta \end{bmatrix}^T \begin{bmatrix} 0 & \bar{s}\bar{E} \\ \bar{E}^T\bar{s} & \bar{H}_E \end{bmatrix} \begin{bmatrix} \frac{\Upsilon(y)}{\phi_1'(e_1)}\Psi \\ \zeta \end{bmatrix}}_{\omega\left(\frac{\Upsilon(y)}{\phi_1'(e_1)}\Psi,\zeta\right)} \geq 0. \tag{B.7}$$

The inequality (B.7) is a general representation of the following cases (B.8) and (B.9) regarding different values for parameter $\bar{r}$:

Case $\bar{r} \geq 0$. The following dissipative inequality is obtained from Assumption A3-ii) with $\bar{r} \geq 0$

$$\begin{bmatrix} \sqrt{\frac{\Upsilon(y)}{\phi_1'(e_1)}}\Psi \\ \sqrt{\frac{\Upsilon(y)}{\phi_1'(e_1)}}h \end{bmatrix}^T \begin{bmatrix} 0 & \bar{s} \\ \bar{s} & \bar{r} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{\Upsilon(y)}{\phi_1'(e_1)}}\Psi \\ \sqrt{\frac{\Upsilon(y)}{\phi_1'(e_1)}}h \end{bmatrix} \geq 0$$

When h takes the value of $\bar{z} = (e_2 + k_4\phi_1(e_1)) = \bar{E}\zeta$ defined in (B.4), and as the function ϕ_1 satisfies $\phi_1'(e_1) \geq \mu_2 > 0$ with $e_1 \neq 0$, the following inequality is satisfied

$$\underbrace{\begin{bmatrix} \frac{\Upsilon}{\phi_1'(e_1)}\Psi \\ \zeta \end{bmatrix}^T \begin{bmatrix} 0 & \bar{s}\bar{E} \\ \bar{E}^T\bar{s} & \bar{E}^T\frac{\bar{r}d_2}{\mu_2}\bar{E} \end{bmatrix} \begin{bmatrix} \frac{\Upsilon(y)}{\phi_1'(e_1)}\Psi \\ \zeta \end{bmatrix}}_{\omega\left(\frac{\Upsilon(y)}{\phi_1'(e_1)}\Psi,\zeta\right)} \geq 0. \tag{B.8}$$

Case $\bar{r} < 0$. The following dissipative inequality is obtained from Assumption A3-ii) with $\bar{r} < 0$

$$\begin{bmatrix} \sqrt{\frac{\Upsilon(y)}{\phi_1'(e_1)}}\Psi \\ \sqrt{\frac{\Upsilon(y)}{\phi_1'(e_1)}}h \end{bmatrix}^T \begin{bmatrix} 0 & \bar{s} \\ \bar{s} & \bar{r} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{\Upsilon(y)}{\phi_1'(e_1)}}\Psi \\ \sqrt{\frac{\Upsilon(y)}{\phi_1'(e_1)}}h \end{bmatrix} \geq 0$$

When h takes the value of $\bar{z} = (e_2 + k_4\phi_1(e_1)) = \bar{E}\zeta$ defined in (B.4), and since the function ϕ_1 satisfies $\phi_1'(e_1) \geq \mu_2 > 0$ with $e_1 \neq 0$, the following inequality is satisfied

$$\underbrace{\begin{bmatrix} \frac{\Upsilon(y)}{\phi_1'(e_1)}\Psi \\ \zeta \end{bmatrix}^T \begin{bmatrix} 0 & \bar{s}\bar{E} \\ \bar{E}^T\bar{s} & 0 \end{bmatrix} \begin{bmatrix} \frac{\Upsilon(y)}{\phi_1'(e_1)}\Psi \\ \zeta \end{bmatrix}}_{\omega\left(\frac{\Upsilon(y)}{\phi_1'(e_1)}\Psi,\zeta\right)} \geq 0. \tag{B.9}$$

Let's find a bound for the term $2\zeta^T PB\Psi$, this is going to be used in the analysis of the time derivative of $V(e)$.

$$\begin{aligned}
2\zeta^T PB\Upsilon(y)\Psi &= 2\phi_1'(e_1)\zeta \begin{bmatrix} -\theta_3\bar{s}k_4 \\ -\theta_3\bar{s} \end{bmatrix} \frac{\Upsilon(y)\Psi}{\phi_1'(e_1)}, \\
&= \theta_3\phi_1'(e_1) \begin{bmatrix} \frac{\Upsilon(y)\Psi}{\phi_1'(e_1)} \\ \zeta \end{bmatrix}^T \begin{bmatrix} 0 & -\bar{s}\bar{E} \\ -\bar{s}\bar{E}^T & 0 \end{bmatrix} \begin{bmatrix} \frac{\Upsilon(y)\Psi}{\phi_1'(e_1)} \\ \zeta \end{bmatrix}, \\
&= \theta_3\phi_1'(e_1) \left(\begin{bmatrix} \frac{\Upsilon(y)\Psi}{\phi_1'(e_1)} \\ \zeta \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & \bar{H}_E \end{bmatrix} \begin{bmatrix} \frac{\Upsilon(y)\Psi}{\phi_1'(e_1)} \\ \zeta \end{bmatrix} + \right. \\
&\qquad \left. - \begin{bmatrix} \frac{\Upsilon(y)\Psi}{\phi_1'(e_1)} \\ \zeta \end{bmatrix}^T \begin{bmatrix} 0 & \bar{s}\bar{E} \\ \bar{s}\bar{E}^T & \bar{H}_E \end{bmatrix} \begin{bmatrix} \frac{\Upsilon(y)\Psi}{\phi_1'(e_1)} \\ \zeta \end{bmatrix} \right), \\
&\leq \phi_1'(e_1)\theta_3\zeta^T\bar{H}_E\zeta,
\end{aligned} \tag{B.10}$$

the last inequality is obtained by (B.7), where $\phi_1'(e_1)\theta_3 > 0$.

For $e_1 \neq 0$, we have

$$\begin{aligned}
\dot{\zeta} &= \begin{bmatrix} \phi_1'(e_1)\left(e_2 - k_1\phi_1(e_1)\right) \\ -\phi_1'(e_1)\phi_1(e_1)k_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \Upsilon(y)\left(\Phi + \Psi - w\right) \end{bmatrix}, \\
&= \phi_1'(e_1)\left(\begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix} \zeta + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Upsilon(y)\frac{\Phi + \Psi - w}{\phi_1'(e_1)} \right), \\
&= \phi_1'(e_1)\left(A\zeta + B\Upsilon(y)\frac{\Phi + \Psi - w}{\phi_1'(e_1)} \right).
\end{aligned} \tag{B.11}$$

Deriving $V(e) = \zeta^T P\zeta$ along of trajectories of e using (B.11) when $e_1 \neq 0$

$$\begin{aligned}
\dot{V}(e(t)) &= \phi_1'(e_1)\zeta^T\left(A^T P + PA\right)\zeta + \\
&\quad + \phi_1'(e_1)\left(\frac{\Upsilon(y)\left(\Phi + \Psi - w\right)}{\phi'(e_1)} B^T P\zeta + \zeta^T PB\frac{\Upsilon(y)\left(\Phi + \Psi - w\right)}{\phi'(e_1)} \right), \\
&= \phi_1'(e_1) \begin{bmatrix} \zeta \\ \frac{\Upsilon(y)}{\phi_1'(e_1)}\Phi \\ \frac{-\Upsilon(y)w}{\phi_1'(e_1)} \end{bmatrix}^T \begin{bmatrix} A^T P + PA & \star & \star \\ B^T P & 0 & \star \\ B^T P & 0 & 0 \end{bmatrix} \begin{bmatrix} \zeta \\ \frac{\Upsilon(y)}{\phi_1'(e_1)}\Phi \\ \frac{-\Upsilon(y)w}{\phi_1'(e_1)} \end{bmatrix} + \\
&\qquad + 2\zeta^T PB\Upsilon(y)\Psi,
\end{aligned}$$

$$
\begin{aligned}
&\le \phi_1'(e_1) \begin{bmatrix} \zeta \\ \frac{\Upsilon(y)}{\phi_1'(e_1)}\Phi \\ \frac{-\Upsilon(y)w}{\phi_1'(e_1)} \end{bmatrix}^T \begin{bmatrix} A^TP + PA + \theta_3 \bar{H}_E & \star & \star \\ B^TP & 0 & \star \\ B^TP & 0 & 0 \end{bmatrix} \begin{bmatrix} \zeta \\ \frac{\Upsilon(y)}{\phi_1'(e_1)}\Phi \\ \frac{-\Upsilon(y)w}{\phi_1'(e_1)} \end{bmatrix} + \\
&\qquad + \theta_1 \phi_1'(e_1)\omega\left(\frac{\Upsilon(y)}{\phi_1'(e_1)}\Phi, \zeta\right) + \theta_2 \phi_1'(e_1)\omega\left(\frac{-\Upsilon(y)w}{\phi_1'(e_1)}, \zeta\right), \\
&\le -\frac{\epsilon}{\lambda_{\max}(P)}\phi_1'(e_1)\zeta^T P\zeta, \\
&= -\frac{\epsilon\mu_1}{2\lambda_M(P)}\frac{1}{|e_1|^{1/2}}V - \frac{\epsilon\mu_2}{\lambda_M(P)}V, \\
&\le -\frac{\epsilon\mu_1^2\lambda_m^{1/2}(P)}{2\lambda_M(P)}V^{1/2} - \frac{\epsilon\mu_2}{\lambda_M(P)}V.
\end{aligned}
$$

The first inequality is obtained from the inequalities (B.6) and (B.5), and the second one is obtained from (3.16). Note that the trajectories of the estimation error dynamics cannot stay in the set $S = \{(e_1, e_2) \in \mathbb{R}^2 \setminus \{0\} | e_1 = 0\}$. This means that V is a continuously decreasing function and using the Lyapunov's Theorem for Differential Inclusions [Deimling, 1992; Prop. 14.1 p. 205] (that does not require differentiability of the Lyapunov function). Since the solution of the differential equation $\dot{v} = -\gamma_1 v^{1/2} - \gamma_2 v$ is given by $v(t) = \exp(-\gamma_2 t)\left[v_0^{1/2} + \frac{\gamma_1}{\gamma_2}(1 - \exp(\frac{\gamma_2}{2}t))\right]$, one can conclude that the equilibrium point $(e_1, e_2) = 0$ is reached in finite time from every initial condition. □

Appendix C

Dissipative approach to global SM observers design for 2-DOF mechanical systems: result proofs

C.1 Proof of Lemma 3 (Page 39)

The feasibility of the matrix inequalities (4.10) is proved as:
$i)$ Choose the parameters k_{oi}, $l_{oi} > 0$, for $i = 1, 2$, this ensures that A_i and B_i are Hurwitz matrices.
$ii)$ Find the matrices P_1, P_2 from Lyapunov inequalities $\Pi_{P_1} < 0, \Pi_{P_2} < 0$ and (4.10a), where $\ell = 1$.
$iii)$ Through Schur complement the inequality (4.10b) is equivalent to

$$-\begin{bmatrix} \gamma_1\mu_{m2}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_2\mu_{m2}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_3\mu_{m1}^4\ell^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_4\mu_{m1}^4\ell^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_5\mu_{m2}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_6\mu_{m2}^2 \end{bmatrix} - F^T \begin{bmatrix} \Pi_{P_1}^{-1} & 0 \\ 0 & \Pi_{P_2}^{-1} \end{bmatrix} F < 0,$$

where $F = \begin{bmatrix} P_1B & 0 & P_1B & 0 & P_1B & 0 \\ 0 & P_2B & 0 & P_2B & 0 & P_2B \end{bmatrix}$.
$iv)$ For sufficiently large parameters μ_{m1}, μ_{m2}, the feasibility of (4.10b) is ensured. □

For the proof of Theorem 5, a definition and some previous results are required, which are given as follows.

Definition 3 (Rocha-Cózatl and Moreno, 2011). *A time-varying nonlinearity $\gamma : [0, \infty) \times \mathbb{R}^p \to \mathbb{R}^m$ with $\gamma(t, \upsilon)$, piecewise continuous in t, locally Lipschitz in υ and $\gamma(t, 0) = 0$, is called $\{Q, S, R\}$-dissipative if for each $t \geq 0$ and $\upsilon \in \mathbb{R}^p$ the following*

$$\omega(\gamma(t,\upsilon),\upsilon) = \begin{bmatrix} \gamma(t,\upsilon) \\ \upsilon \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \gamma(t,\upsilon) \\ \upsilon \end{bmatrix} \geq 0, \qquad \text{(C.1)}$$

where $Q = Q^T \in \mathbb{R}^{m\times m}$, $S \in \mathbb{R}^{m\times p}$, $R = R^T \in \mathbb{R}^{p\times p}$, is satisfied. ⋄

Lemma 8. *If a nonlinearity $\gamma(t, \upsilon)$ is $\{Q, S, R\}-$dissipative with Q a negative definite matrix, then there exists a matrix $\tilde{R}$ such that $\gamma(t, \upsilon)$ is $\{Q, 0, \tilde{R}\}-$dissipative.* ▲

Proof of Lemma 8. As the non linearity $\gamma := \gamma(t, \upsilon)$ is $\{Q, S, R\}-$dissipative where $Q < 0$, it is satisfied that

$$\begin{bmatrix} \gamma \\ \upsilon \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \gamma \\ \upsilon \end{bmatrix} = \gamma^T Q\gamma + \gamma^T S\upsilon + \upsilon^T S^T\gamma + \upsilon^T R\upsilon \geq 0.$$

From which we get the following inequalities:

$$\gamma^T S\upsilon + \upsilon^T S^T\gamma + \upsilon^T R\upsilon \geq -\gamma^T Q\gamma \geq -\lambda_M(Q)\|\gamma\|^2, \tag{C.2}$$

$$2\|\gamma\|\|\upsilon\| \left\{\lambda_M(S^T S)\right\}^{1/2} + \lambda_M(R)\|\upsilon\|^2 \geq -\lambda_M(Q)\|\gamma\|^2,$$

where the matrix property $\lambda_m(Q)\|\gamma\|^2 \leq \gamma^T Q\gamma \leq \lambda_M(Q)\|\gamma\|^2$ is used.

Consider the auxiliary inequality $bxy + cy^2 \geq ax^2$, where $a, b, c > 0$, $x, y \geq 0$. Then from $cy^2 \geq -bxy + ax^2$ by completing squares one gets $\left(c/a + b^2/(4a^2)\right) y^2 \geq x^2 - (b/a)xy + (b/(2a))^2 y^2 = (x - (b/2a)y)^2$. Consequently, the inequality

$$\left(\sqrt{c/a + b^2/(4a^2)} + (b/2a)\right) y \geq x,$$

is obtained. Now substituting $x = \|\gamma\|$, $y = \|\upsilon\|$, $a = -\lambda_M(Q)$, $b = 2\left\{\lambda_M(S^T S)\right\}^{1/2}$, $c = \lambda_M(R)$ one gets

$$\left(\sqrt{-\frac{\lambda_M(R)}{\lambda_M(Q)} + \frac{\lambda_M(S^T S)}{\lambda_M^2(Q)}} - \frac{\{\lambda_M(S^T S)\}^{1/2}}{\lambda_M(Q)}\right) \|\upsilon\| \geq \|\gamma\|,$$
$$\frac{\sqrt{-\lambda_M(R)\lambda_M(Q) + \lambda_M(S^T S)} + \{\lambda_M(S^T S)\}^{1/2}}{-\lambda_M(Q)} \|\upsilon\| \geq \|\gamma\|. \tag{C.3}$$

According to the inequality $\frac{1}{\lambda_m(Q)}\gamma^T Q\gamma \leq \|\gamma\|^2$ we obtain

$$\tilde{R} = \frac{-\lambda_m(Q)}{\lambda_M^2(Q)} \left(\sqrt{-\lambda_M(R)\lambda_M(Q) + \lambda_M(S^T S)} + \sqrt{\lambda_M(S^T S)}\right)^2 I_p,$$

where I_p is the identity matrix of size p. □

Lemma 9. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}$ a function such that there exist non negative parameters l_γ, L_γ, and*

$$l_\gamma \leq \gamma(t) \leq L_\gamma, \tag{C.4}$$

is satisfied, then

a) the equality

$$\gamma(t) = h_1 \cdot (l_\gamma - a) + h_2 \cdot (L_\gamma + b), \tag{C.5}$$

is satisfied for $h_1 := \frac{L_\gamma + b - \gamma(t)}{L_\gamma - l_\gamma + a + b}$, $h_2 := 1 - h_1$ with a, b satisfying $a, b \geq 0$ and $l_\gamma - a \geq 0$. The functions h_1 and h_2 also satisfy

$$\frac{b}{L_\gamma - l_\gamma + a + b} \leq h_1 \leq \frac{L_\gamma - l_\gamma + b}{L_\gamma - l_\gamma + a + b}, \tag{C.6}$$
$$\frac{a}{L_\gamma - l_\gamma + a + b} \leq h_2 \leq \frac{L_\gamma - l_\gamma + a}{L_\gamma - l_\gamma + a + b}.$$

b) and the equality

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & \gamma(t) \end{bmatrix} = h_1 \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & l_\gamma - a \end{bmatrix} + h_2 \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & L_\gamma + b \end{bmatrix}, \tag{C.7}$$

is satisfied, where a_{11}, a_{12}, $a_{21} \in \mathbb{R}$.

▲

Proof of Lemma 9. *a*) Using (C.4) one can verify the inequalities (C.6) from the functions $h_1 := \frac{L_\gamma+b-\gamma(t)}{L_\gamma-l_\gamma+a+b}$ and $h_2 = \frac{a-l_\gamma+\gamma(t)}{L_\gamma-l_\gamma+a+b}$. For inequality (C.5) consider the following

$$\begin{aligned} h_1 \cdot (l_\gamma - a) + h_2 \cdot (L_\gamma + b) =& \frac{L_\gamma + b - \gamma(t)}{L_\gamma - l_\gamma + a + b} \cdot (l_\gamma - a) + \frac{a - l_\gamma + \gamma(t)}{L_\gamma - l_\gamma + a + b} \cdot (L_\gamma + b) \\ =& \frac{\gamma(t)(L_\gamma + b - l_\gamma + a)}{L_\gamma - l_\gamma + a + b} = \gamma(t). \end{aligned}$$

b) The equality (C.7) is obtained from (C.5) using the condition $h_1 + h_2 = 1$. □

C.2 Proof of Theorem 5 (Page 40)

To illustrate better the dependence of variables, consider the diffeomorphism (4.6) as

$$T(x,z) = \begin{bmatrix} T_1(x) \\ T_2(x,z) \end{bmatrix} \tag{C.8}$$

where $T_1(x) = \begin{bmatrix} x_1 + \int_0^{x_2} \frac{m_{12}(s)}{m_{11}} ds \\ x_2 \end{bmatrix}$, $T_2(x,z) = \begin{bmatrix} m_{11}z_1 + m_{12}(x_2)z_2 \\ \alpha(x_2)z_2 \end{bmatrix}$. From the time derivative of (4.6) and the system (4.5) one obtains

$$\dot{\theta} = \frac{\partial T_1(x)}{\partial x} z = \frac{\partial T_1(x)}{\partial x} \Upsilon_{\theta_2} w, \tag{C.9}$$

$$\begin{aligned} \dot{w} =& \left[\frac{\partial T_2(x,z)}{\partial x} - \frac{\partial T_2(x,z)}{\partial z} M^{-1}(x_2) C(x,z) \right] z + \\ & + \frac{\partial T_2(x,z)}{\partial z} M^{-1}(x_2) \left[v - Hz - \psi + \delta \right], \\ =& \left[\frac{\partial T_2(x,z)}{\partial x} - \frac{\partial T_2(x,z)}{\partial z} M^{-1}(x_2) C(x,z) \right] \Upsilon_{\theta_2} z + \\ & + \frac{\partial T_2(x,z)}{\partial z} M^{-1}(x_2) \left[v - H\Upsilon_{\theta_2} z - \psi + \delta \right]. \end{aligned} \tag{C.10}$$

with $\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$, $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and $\Upsilon_{\theta_2} = \begin{bmatrix} \frac{1}{m_{11}} & -\frac{m_{12}(\theta_2)}{m_{11}\alpha(\theta_2)} \\ 0 & \frac{1}{\alpha(\theta_2)} \end{bmatrix}$.

Note that $M^{-1}(x_2) = \frac{1}{det(M(x_2))} \begin{bmatrix} m_{22}(x_2) & -m_{12}(x_2) \\ -m_{12}(x_2) & m_{11}(x_2) \end{bmatrix}$, $\frac{\partial T_1(x)}{\partial x} = \begin{bmatrix} 1 & \frac{m_{12}(x_2)}{m_{11}} \\ 0 & 1 \end{bmatrix}$, $\frac{\partial T_2(x,z)}{\partial x} = \begin{bmatrix} 0 & m'_{12}(x_2)z_2 \\ 0 & \alpha'(x_2)z_2 \end{bmatrix}$, $\frac{\partial T_2(x,z)}{\partial z} = \begin{bmatrix} m_{11} & m_{12}(x_2) \\ 0 & \alpha(x_2) \end{bmatrix}$,

$\frac{\partial T_2(x,z)}{\partial z} M^{-1}(x_2) = \begin{bmatrix} 1 & 0 \\ -\frac{m_{12}(x_2)\alpha(x_2)}{det(M(x_2))} & \frac{m_{11}\alpha(x_2)}{det(M(x_2))} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{m_{12}(x_2)}{m_{11}\alpha(x_2)} & \frac{m_{11}}{m_{11}\alpha(x_2)} \end{bmatrix}$,

where $\alpha(x_2) = \sqrt{\frac{det(M(x_2))}{m_{11}}}$ and its derivative w.r.t. x_2 is given by

$$\alpha'_2(x_2) = \frac{m_{11}m'_{22}(x_2) - 2m_{12}(x_2)m'_{12}(x_2)}{2\alpha(x_2)m_{11}}. \tag{C.11}$$

Thus, with the previous derivative (C.11) the equality $\frac{\partial T_2(x,z)}{\partial x} - \frac{\partial T_2(x,z)}{\partial z} M^{-1}(x_2) C(x,z) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, is obtained. From (C.9) one can obtain

$$
\begin{aligned}
\dot{\theta}_1 &= \left(\frac{\partial T_1(x)}{\partial x} \Upsilon_{\theta_2} w \right)_{11} = \frac{w_1}{m_{11}}, \\
\dot{w}_1 &= v_1 - (H \Upsilon_{\theta_2} w)_{11} - \psi_1 + \delta_1, \\
\dot{\theta}_2 &= \left(\frac{\partial T_1(x)}{\partial x} \Upsilon_{\theta_2} w \right)_{21} = \frac{w_2}{\alpha(\theta_2)}, \\
\dot{w}_2 &= \left(\frac{\partial T_2(x,z)}{\partial z} M^{-1}(x_2) \left[v - H \Upsilon_{\theta_2} z - \psi + \delta \right] \right)_{21} \\
&= \frac{m_{11}[-(H\Upsilon_{\theta_2})_{21} w_1 - (H\Upsilon_{\theta_2})_{22} w_2]}{m_{11}\alpha(\theta_2)} + \\
&\quad - \frac{m_{12}(\theta_2)[-(H\Upsilon_{\theta_2})_{11} w_1 - (H\Upsilon_{\theta_2})_{12} w_2]}{m_{11}\alpha(\theta_2)} + \\
&\quad + \frac{m_{11}(-\psi_2(x, \Upsilon_{\theta_2} w)) - m_{12}(\theta_2)(-\psi_1(x, \Upsilon_{\theta_2} w))}{m_{11}\alpha(\theta_2)} \\
&\quad + \frac{m_{11} v_2 - m_{12}(\theta_2) v_1 + m_{11}\delta_2 - m_{12}(\theta_2)\delta_1}{m_{11}\alpha(\theta_2)}.
\end{aligned}
$$

The dynamics of the estimation error between the systems (4.8) and (4.9a) is given by

$$
\begin{aligned}
\dot{e}_{\theta_1} &= \frac{e_{w_1}}{m_{11}} - \ell k_{o1} \phi_{11}(e_{\theta_1}), \\
\dot{e}_{w_1} &= -(H\Upsilon_{\theta_2})_{11} e_{w_1} - (H\Upsilon_{\theta_2})_{12} e_{w_2} + \Psi_1 - \delta_1 - \ell^2 k_{o2} \phi_{12}(\theta_1), \\
\dot{e}_{\theta_2} &= \frac{e_{w_2} - \ell l_{o1} \phi_{21}(e_{\theta_2})}{\alpha(\theta_2)}, \\
\dot{e}_{w_2} &= \frac{m_{11}[-(H\Upsilon_{\theta_2})_{21} e_{w_1} - (H\Upsilon_{\theta_2})_{22} e_{w_2}]}{m_{11}\alpha(\theta_2)} + \\
&\quad - \frac{m_{12}(\theta_2)[-(H\Upsilon_{\theta_2})_{11} e_{w_1} - (H\Upsilon_{\theta_2})_{12} e_{w_2}]}{m_{11}\alpha(\theta_2)} + \\
&\quad + \frac{m_{11}\Psi_2 - m_{12}(\theta_2)\Psi_1}{m_{11}\alpha(\theta_2)} - \frac{m_{11}\delta_2 - m_{12}(\theta_2)\delta_1}{m_{11}\alpha(\theta_2)} - \frac{\ell^2 l_{o2} \phi_{22}(e_{\theta_2})}{\alpha_2(\theta_2)},
\end{aligned} \tag{C.12}
$$

where $e_{\theta_1} = \hat{\theta}_1 - \theta_1$, $e_{w_1} = \hat{w}_1 - w_1$, $e_{\theta_2} = \hat{\theta}_2 - \theta_2$, $e_{w_2} = \hat{w}_2 - w_2$, $\Psi_1 := \psi_1(x, \Upsilon_{\theta_2} w) - \psi_1(x, \Upsilon_{\theta_2} w + \Upsilon_{\theta_2}(e_w + k_{ol}\phi_1(e_\theta)))$, $\Psi_2 := \psi_2(x, \Upsilon_{\theta_2} w) - \psi_2(x, \Upsilon_{\theta_2} w + \Upsilon_{\theta_2}(e_w + k_{ol}\phi_1(e_\theta)))$.

For $\zeta = \begin{bmatrix} \phi_{11}(e_{\theta_1}) \\ e_{w_1} \end{bmatrix}$, one has

$$
\dot{\zeta} = \phi'_{11}(e_{\theta_1}) \begin{bmatrix} -\ell k_{o1} & \frac{1}{m_{11}} \\ -\ell^2 k_{o2} & -\frac{(H\Upsilon_{\theta_2})_{11}}{\phi'_{11}(e_{\theta_1})} \end{bmatrix} \zeta + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} \left(-(H\Upsilon_{\theta_2})_{12} e_{w_2} + \Psi_1 - \delta_1 \right).
$$

The state transformation

$$
\tilde{\zeta} = \ell^2 \Delta_\ell^{-1} \zeta, \tag{C.13}
$$

leads to

$$\dot{\tilde{\zeta}} = \ell\phi'_{11}(e_{\theta_1})\begin{bmatrix} -k_{o1} & \frac{1}{m_{11}} \\ -k_{o2} & -\frac{(H\Upsilon_{\theta_2})_{11}}{\ell\phi'_{11}(e_{\theta_1})} \end{bmatrix}\tilde{\zeta} + B(-(H\Upsilon_{\theta_2})_{12}\tilde{e}_{w_2} + \Psi_1 - \delta_1).$$

From Assumption P-3, one obtains

$$0 \leq \frac{(H\Upsilon_{\theta_2})_{11}}{\ell\phi'_{11}(e_{\theta_1})} \leq \frac{\overline{(H\Upsilon_{\theta_2})_{11}}}{\ell\mu_{12}}, \tag{C.14}$$

Now, from Lemma 9 one can get

$$\begin{bmatrix} -k_{o1} & \frac{1}{m_{11}} \\ -k_{o2} & -\frac{(H\Upsilon_{\theta_2})_{11}}{\ell\phi'_{11}(e_{\theta_1})} \end{bmatrix} = h_{A1}\cdot A_1 + h_{A2}\cdot A_2, \tag{C.15}$$

where $\underline{h_{A1}} = \frac{b_1}{\frac{\overline{(H\Upsilon_{\theta_2})_{11}}}{\ell\mu_{12}}+b_1} = \frac{b_1\ell\mu_{12}}{\overline{(H\Upsilon_{\theta_2})_{11}}+b_1\ell\mu_{12}}$ y $\underline{h_{A2}} = 0$.

Define $\vartheta_0 := \frac{m_{11}(H\Upsilon_{\theta_2})_{22} - m_{12}(\theta_2)(H\Upsilon_{\theta_2})_{12}}{m_{11}\phi'_{21}(e_{\theta_2})}$ and $\eta = \begin{bmatrix}\phi_{21}(e_{\theta_2}) \\ e_{w_2}\end{bmatrix}$, where the time derivative of η is given as

$$\begin{aligned}\dot{\eta} =& \frac{\phi'_{21}(e_{\theta_2})}{\alpha(\theta_2)}\begin{bmatrix} -\ell l_{o1} & 1 \\ -\ell^2 l_{o2} & \vartheta_0 \end{bmatrix}\eta - B\frac{m_{11}\delta_2 - m_{12}(\theta_2)\delta_1}{m_{11}\alpha(\theta_2)} \\ &+ B\frac{-m_{11}(H\Upsilon_{\theta_2})_{21} + m_{12}(\theta_2)(H\Upsilon_{\theta_2})_{11}}{m_{11}\alpha(\theta_2)}e_{w_1} \\ &+ B\frac{m_{11}\Psi_2 - m_{12}(\theta_2)\Psi_1}{m_{11}\alpha(\theta_2)}.\end{aligned} \tag{C.16}$$

From $\tilde{\eta} = \ell^2\Delta_\ell^{-1}\eta$ it follows

$$\begin{aligned}\dot{\tilde{\eta}} =& \frac{\ell\phi'_{21}(e_{\theta_2})}{\alpha(\theta_2)}\begin{bmatrix} -l_{o1} & 1 \\ -l_{o2} & -\frac{\vartheta_0}{\ell} \end{bmatrix}\tilde{\eta} - B\frac{m_{11}\delta_2 - m_{12}(\theta_2)\delta_1}{m_{11}\alpha(\theta_2)} + \\ &+ B\frac{-m_{11}(H\Upsilon_{\theta_2})_{21} + m_{12}(\theta_2)(H\Upsilon_{\theta_2})_{11}}{\alpha(\theta_2)m_{11}}\tilde{e}_{w_1} \\ &+ B\frac{m_{11}\Psi_2 - m_{12}(\theta_2)\Psi_1}{m_{11}\alpha(\theta_2)}.\end{aligned}$$

From the inequality

$$0 \leq \frac{\vartheta_0}{\ell} \leq \frac{m_{11}\overline{(H\Upsilon_{\theta_2})_{22}} + \overline{m_{12}(\theta_2)(H\Upsilon_{\theta_2})_{12}}}{\ell m_{11}\mu_{22}}, \tag{C.17}$$

obtained by Assumption P-3, and applying Lemma 9 one has

$$\begin{bmatrix} -l_{o1} & 1 \\ -l_{o2} & -\frac{\vartheta_0}{\ell} \end{bmatrix} = h_{B1}\cdot B_1 + h_{B2}\cdot B_2, \tag{C.18}$$

where $\underline{h_{B1}} = \frac{b_2}{\frac{m_{11}\overline{(H\Upsilon_{\theta_2})_{22}} + \overline{m_{12}(\theta_2)(H\Upsilon_{\theta_2})_{12}}}{\ell m_{11}\mu_{22}}+b_2}$ and $\underline{h_{B2}} = 0$.

Consider the following Lyapunov function candidate

$$V(e) = \begin{bmatrix} \tilde{\zeta} \\ \tilde{\eta} \end{bmatrix}^T \underbrace{\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}}_{P} \begin{bmatrix} \tilde{\zeta} \\ \tilde{\eta} \end{bmatrix},$$

where P_1 (P_2) is a common Lyapunov function for A_1 and A_2 (B_1 y B_2) respectively, i.e. the inequalities

$$\begin{aligned} &A_1^T P_1 + P_1 A_1 < 0 \text{ and } A_2^T P_1 + P_1 A_2 < 0, \\ &B_1^T P_2 + P_2 B_1 < 0 \text{ and } B_2^T P_2 + P_2 B_2 < 0, \end{aligned} \tag{C.19}$$

are satisfied.

The time derivative of V is given as

$$\begin{aligned} \dot{V} =& \ell \phi_{11}'(e_{\theta_1}) \tilde{\zeta}^T \left(h_{A1} A_1^{P_1} + h_{A2} A_2^{P_1} \right) \tilde{\zeta} + \\ &+ 2\tilde{\zeta}^T P_1 B (-(H\Upsilon_{\theta_2})_{12} \tilde{e}_{w_2} + \Psi_1 - \delta_1) + \\ &+ \frac{\ell \phi_{21}'(e_{\theta_2})}{\alpha(\theta_2)} \tilde{\eta}^T \left(h_{B1} B_1^{P_2} + h_{B2} B_2^{P_2} \right) \tilde{\eta} + \\ &+ 2\tilde{\eta}^T P_2 B \frac{-m_{11}(H\Upsilon_{\theta_2})_{21} + m_{12}(\theta_2)(H\Upsilon_{\theta_2})_{11}}{m_{11}\alpha(\theta_2)} \tilde{e}_{w_1} + \\ &+ 2\tilde{\eta}^T P_2 B \frac{m_{11}\Psi_2 - m_{12}(\theta_2)\Psi_1}{m_{11}\alpha(\theta_2)} + \\ &- 2\tilde{\eta}^T P_2 B \frac{m_{11}\delta_2 - m_{12}(\theta_2)\delta_1}{m_{11}\alpha(\theta_2)}, \end{aligned}$$

where $A_1^{P_1} = A_1^T P_1 + P_1 A_1$, $A_2^{P_1} = A_2^T P_1 + P_1 A_2$, $B_1^{P_2} = B_1^T P_2 + P_2 B_1$, $B_2^{P_2} = B_2^T P_2 + P_2 B_2$.

From which one obtains:

$$\begin{aligned}
\dot{V} \leq\; & \ell\phi_{11}'(e_{\theta_1})\underline{h_{A1}}\tilde{\zeta}^T A_1^{P_1}\tilde{\zeta} - 2\tilde{\zeta}^T P_1 B((H\Upsilon_{\theta_2})_{12}\tilde{e}_{w_2} - \Psi_1 + \delta_1) \\
& + \frac{\ell\phi_{21}'(e_{\theta_2})}{\alpha(\theta_2)}\underline{h_{B1}}\tilde{\eta}^T B_1^{P_2}\tilde{\eta} - 2\tilde{\eta}^T P_2 B\frac{m_{11}\delta_2 - m_{12}(\theta_2)\delta_1}{m_{11}\alpha(\theta_2)} \\
& + 2\tilde{\eta}^T P_2 B\frac{-m_{11}(H\Upsilon_{\theta_2})_{21} + m_{12}(\theta_2)(H\Upsilon_{\theta_2})_{11}}{m_{11}\alpha(\theta_2)}\tilde{e}_{w_1} \\
& + 2\tilde{\eta}^T P_2 B\frac{m_{11}\Psi_2 - m_{12}(\theta_2)\Psi_1}{m_{11}\alpha(\theta_2)}, \qquad \text{(C.20)} \\
\leq & \beta\ell\left[\underline{h_{A1}}\tilde{\zeta}^T A_1^{P_1}\tilde{\zeta} + \frac{\underline{h_{B1}}}{\alpha(\theta_2)}\tilde{\eta}^T B_1^{P_2}\tilde{\eta}\right] + \\
& + 2\tilde{\zeta}^T P_1 B(-(H\Upsilon_{\theta_2})_{12}\tilde{e}_{w_2} + \Psi_1 - \delta_1) + \\
& + 2\tilde{\eta}^T P_2 B\frac{-m_{11}(H\Upsilon_{\theta_2})_{21} + m_{12}(\theta_2)(H\Upsilon_{\theta_2})_{11}}{m_{11}\alpha(\theta_2)}\tilde{e}_{w_1} \\
& + 2\tilde{\eta}^T P_2 B\left(\frac{m_{11}\Psi_2 - m_{12}(\theta_2)\Psi_1}{m_{11}\alpha(\theta_2)} - \frac{m_{11}\delta_2 - m_{12}(\theta_2)\delta_1}{m_{11}\alpha(\theta_2)}\right), \\
= & \beta X^T \begin{bmatrix}
\ell\underline{h_{A1}}A_1^{P_1} & \star & \star & \star & \star & \star & \star & \star \\
0 & \frac{\ell\underline{h_{B1}}}{\underline{\alpha(\theta_2)}}B_1^{P_2} & \star & \star & \star & \star & \star & \star \\
B^T P_1 & 0 & 0 & \star & \star & \star & \star & \star \\
0 & B^T P_2 & 0 & 0 & \star & \star & \star & \star \\
B^T P_1 & 0 & 0 & 0 & 0 & \star & \star & \star \\
0 & B^T P_2 & 0 & 0 & 0 & 0 & \star & \star \\
B^T P_1 & 0 & 0 & 0 & 0 & 0 & 0 & \star \\
0 & B^T P_2 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} X, \qquad \text{(C.21)}
\end{aligned}$$

where $\beta = \frac{\mu_{m1}}{|e_{\theta_1}|^{1/2} + |e_{\theta_2}|^{1/2}} + \mu_{m2}$, $\mu_{mj} = \min\{\mu_{1j}, \mu_{2j}\}$ for $j = 1, 2$, and

$$X = \begin{bmatrix}
\tilde{\zeta} \\
\tilde{\eta} \\
-\frac{(H\Upsilon_{\theta_2})_{12}}{\beta}\tilde{e}_{w_2} \\
\frac{-m_{11}(H\Upsilon_{\theta_2})_{21} + m_{12}(\theta_2)(H\Upsilon_{\theta_2})_{11}}{\beta\alpha(\theta_2)m_{11}}\tilde{e}_{w_1} \\
-\frac{\delta_1}{\beta} \\
-\frac{m_{11}\delta_2 - m_{12}(\theta_2)\delta_1}{\beta m_{11}\alpha(\theta_2)} \\
\frac{\Psi_1}{\beta} \\
\frac{m_{11}\Psi_2 - m_{12}(\theta_2)\Psi_1}{\beta m_{11}\alpha(\theta_2)}
\end{bmatrix}.$$

Analyzing the following dissipativity properties of non linearities, one can conclude:
$i)$ For $\vartheta_1 := -\frac{(H\Upsilon_{\theta_2})_{12}}{\beta}\tilde{e}_{w_2}$ the following inequality is satisfied

$$|\vartheta_1|^2 \leq \frac{\Gamma_1}{\mu_{m2}^2}\tilde{\eta}^T B B^T \tilde{\eta}, \qquad \text{(C.22)}$$

where $\Gamma_1 = \overline{(H\Upsilon_{\theta_2})_{12}}^2$. Which implies the following dissipativity property

$$\begin{bmatrix} \tilde{\eta} \\ \vartheta_1 \end{bmatrix}^T \begin{bmatrix} \gamma_1\Gamma_1 B B^T & 0 \\ 0 & -\gamma_1\mu_{m2}^2 \end{bmatrix} \begin{bmatrix} \tilde{\eta} \\ \vartheta_1 \end{bmatrix} \geq 0, \qquad \text{(C.23)}$$

for all $\gamma_1 > 0$.
ii) For $\vartheta_2 := \frac{-m_{11}(H\Upsilon_{\theta_2})_{21}+m_{12}(\theta_2)(H\Upsilon_{\theta_2})_{11}}{\beta\alpha(\theta_2)m_{11}}\tilde{e}_{w_1}$, the following inequality is satisfied

$$|\vartheta_2|^2 \leq \frac{\Gamma_2}{\mu_{m2}^2}\tilde{\zeta}^T BB^T\tilde{\zeta}, \tag{C.24}$$

where $\Gamma_2 = \overline{\left(\frac{-m_{11}(H\Upsilon_{\theta_2})_{21}+m_{12}(\theta_2)(H\Upsilon_{\theta_2})_{11}}{\alpha(\theta_2)m_{11}}\right)^2}$. This implies the following dissipativity condition

$$\begin{bmatrix}\tilde{\zeta}\\ \vartheta_2\end{bmatrix}^T \begin{bmatrix}\gamma_2\Gamma_2 BB^T & 0\\ 0 & -\gamma_2\mu_{m2}^2\end{bmatrix}\begin{bmatrix}\tilde{\zeta}\\ \vartheta_2\end{bmatrix} \geq 0, \tag{C.25}$$

for all $\gamma_2 > 0$.
$iii)$ For $-\frac{\delta_1}{\beta}$ the following inequality is satisfied

$$\begin{aligned}\left|-\frac{\delta_1}{\beta}\right|^2 &\leq \left(\frac{|e_{\theta_1}|^{1/2}+|e_{\theta_2}|^{1/2}}{\mu_{m1}}\right)^2 L_{\delta_1}^2,\\ &= \frac{\left(|e_{\theta_1}|^{1/2}+|e_{\theta_2}|^{1/2}\right)^2 L_{\delta_1}^2\|(\ell\phi_{11}(e_{\theta_1}),\ell\phi_{21}(e_{\theta_2}))\|^2}{\mu_{m1}^2\ell^2\|(\phi_{11}(e_{\theta_1}),\phi_{21}(e_{\theta_2}))\|^2},\\ &\leq \frac{4}{\mu_{m1}^4\ell^2}L_{\delta_1}^2\|(\ell\phi_{11}(e_{\theta_1}),\ell\phi_{21}(e_{\theta_2}))\|^2,\end{aligned}$$

i.e.

$$\begin{bmatrix}\tilde{\zeta}\\ \tilde{\eta}\\ -\frac{\delta_1}{\beta}\end{bmatrix}^T \begin{bmatrix}\gamma_3\Gamma_3\tilde{B}^T\tilde{B} & 0 & 0\\ 0 & \gamma_3\Gamma_3\tilde{B}^T\tilde{B} & 0\\ 0 & 0 & -\gamma_3\mu_{m1}^4\ell^2\end{bmatrix}\begin{bmatrix}\tilde{\zeta}\\ \tilde{\eta}\\ -\frac{\delta_1}{\beta}\end{bmatrix} \geq 0, \tag{C.26}$$

for all $\gamma_3 > 0$, where $\tilde{B} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ and $\Gamma_3 = 4L_{\delta_1}^2$.
$iv)$ For $\vartheta_3 = -\frac{m_{11}\delta_2 - m_{12}(\theta_2)\delta_1}{\beta m_{11}\alpha(\theta_2)}$, with an analysis analogous to the previous one, the following is obtained

$$\begin{bmatrix}\tilde{\zeta}\\ \tilde{\eta}\\ \vartheta_3\end{bmatrix}^T \begin{bmatrix}\gamma_4\Gamma_4\tilde{B}^T\tilde{B} & 0 & 0\\ 0 & \gamma_4\Gamma_4\tilde{B}^T\tilde{B} & 0\\ 0 & 0 & -\gamma_4\mu_{m1}^4\ell^2\end{bmatrix}\begin{bmatrix}\tilde{\zeta}\\ \tilde{\eta}\\ \vartheta_3\end{bmatrix} \geq 0, \tag{C.27}$$

for all $\gamma_4 > 0$, where $\Gamma_4 = 4\overline{\left|\frac{m_{11}\delta_2 - m_{12}(\theta_2)\delta_1}{m_{11}\alpha(\theta_2)}\right|^2}$.
$v)$ For the term $\frac{\Psi_1}{\beta}$, from Assumption P-5 and (C.3) with $h = \Upsilon_{\theta_2}\left(e_w + k_l\phi_1(e_\theta)\right)$, one obtains

$$\left(\frac{\Psi_1}{\beta}\right)^2 \leq \frac{\Gamma_5}{\mu_{m2}^2}\begin{bmatrix}\zeta\\ \eta\end{bmatrix}^T\begin{bmatrix}E_k^T E_k & 0\\ 0 & E_l^T E_l\end{bmatrix}\begin{bmatrix}\zeta\\ \eta\end{bmatrix}, \tag{C.28}$$

which is equivalent to the inequality

$$\begin{bmatrix}\tilde{\zeta}\\ \tilde{\eta}\\ \frac{\Psi_1}{\beta}\end{bmatrix}^T \begin{bmatrix}\gamma_5\Gamma_5\ell^4\tilde{\Delta}_1 & 0 & 0\\ 0 & \gamma_5\Gamma_5\ell^4\tilde{\Delta}_1 & 0\\ 0 & 0 & -\gamma_5\mu_{m2}^2\end{bmatrix}\begin{bmatrix}\tilde{\zeta}\\ \tilde{\eta}\\ \frac{\Psi_1}{\beta}\end{bmatrix} \geq 0, \tag{C.29}$$

for all $\gamma_5 > 0$, where $\tilde{\Delta}_1 = \Delta_\ell^{-1}E_k^T E_k\Delta_\ell^{-1}$, $E_k = \begin{bmatrix}k_{o3} & 1\end{bmatrix}$, $E_l = \begin{bmatrix}l_{o3} & 1\end{bmatrix}$ and $\Gamma_5 = \Gamma_{R0}^2\overline{\lambda_{\max}\left(\Upsilon_{\theta_2}^T\Upsilon_{\theta_2}\right)}$.

$vi)$ For the term $\vartheta_4 = \frac{m_{11}\Psi_2 - m_{12}(\theta_2)\Psi_1}{\beta m_{11}\alpha(\theta_2)}$, from Assumption P-5 and (C.3) with $h = \Upsilon_{\theta_2}(e_w + k_l\phi_1(e_\theta))$, one obtains

$$\vartheta_4^2 \leq \frac{\Gamma_6}{\mu_{m2}^2}\begin{bmatrix}\zeta\\ \eta\end{bmatrix}^T \begin{bmatrix}E_k^T E_k & 0\\ 0 & E_l^T E_l\end{bmatrix}\begin{bmatrix}\zeta\\ \eta\end{bmatrix}, \tag{C.30}$$

where $\Gamma_6 = \overline{\left(\frac{\|(m_{11}, m_{12}(\theta_2))\|^2 \Gamma_{R0}^2 \lambda_{\max}\left(\Upsilon_{\theta_2}^T \Upsilon_{\theta_2}\right)}{m_{11}^2\alpha^2(\theta_2)}\right)}$, which is equivalent to the following inequality

$$\begin{bmatrix}\tilde\zeta\\ \tilde\eta\\ \vartheta_4\end{bmatrix}^T \begin{bmatrix}\gamma_6\Gamma_6\ell^4\tilde\Delta_1 & 0 & 0\\ 0 & \gamma_6\Gamma_6\ell^4\tilde\Delta_1 & 0\\ 0 & 0 & -\gamma_6\mu_{m2}^2\end{bmatrix}\begin{bmatrix}\tilde\zeta\\ \tilde\eta\\ \vartheta_4\end{bmatrix} \geq 0, \tag{C.31}$$

for all $\gamma_6 > 0$.

Adding (C.23), (C.25), (C.26), (C.27), (C.29), (C.31) to the inequality (C.20), the following inequality is obtained

$$\dot V \leq \beta X^T \mathscr{D} \mathscr{X},$$

where $\mathscr{D}$ is equal to

$$\begin{bmatrix}
\mathit{\Pi}_{P1} - \epsilon I & \star & \star & \star & \star & \star & \star & \star\\
0 & \mathit{\Pi}_{P2} - \epsilon I & \star & \star & \star & \star & \star & \star\\
B^T P_1 & 0 & -\gamma_1\mu_{m2}^2 & \star & \star & \star & \star & \star\\
0 & B^T P_2 & 0 & -\gamma_2\mu_{m2}^2 & \star & \star & \star & \star\\
B^T P_1 & 0 & 0 & 0 & -\gamma_3\mu_{m1}^4\ell^2 & \star & \star & \star\\
0 & B^T P_2 & 0 & 0 & 0 & -\gamma_4\mu_{m1}^4\ell^2 & \star & \star\\
B^T P_1 & 0 & 0 & 0 & 0 & 0 & -\gamma_5\mu_{m2}^2 & \star\\
0 & B^T P_2 & 0 & 0 & 0 & 0 & 0 & -\gamma_6\mu_{m2}^2
\end{bmatrix}.$$

From the inequality (4.10b) in Lemma 3, the time derivative of V satisfies the next inequality

$$\dot V \leq -\epsilon\beta\|(\tilde\zeta, \tilde\eta)\|^2, \tag{C.32}$$

where

$$\begin{aligned}
\beta = \frac{\mu_{m1}}{|e_{\theta_1}|^{1/2} + |e_{\theta_2}|^{1/2}} + \mu_{m2} &\geq \frac{\mu_{m1}}{|e_{\theta_1}|^{1/2} + |e_{\theta_2}|^{1/2}},\\
&\geq \frac{\mu_{m1}^2}{2\sqrt{\phi_{11}^2(e_{\theta_1}) + \phi_{21}^2(\theta_2)}},\\
&\geq \frac{\mu_{m1}^2\ell}{2\|(\tilde\zeta, \tilde\eta)\|}.
\end{aligned}$$

Finally, $\dot V \leq -\frac{\epsilon\mu_{m1}^2\ell}{2}\|(\tilde\zeta, \tilde\eta)\| \leq -\frac{\epsilon\mu_{m1}^2\ell}{2\sqrt{\lambda_{\max}(P)}}V^{1/2}$.

Note that the trajectories of the estimation error dynamics cannot remain in the set $S = \{(e_{\theta_1}, e_{\theta_2}, e_{w_1}, e_{w_2}) \in \mathbb{R}^4 \setminus \{0\} | e_{\theta_1} = e_{\theta_2} = 0\}$. This implies that the derivative of V along trajectories of (C.12) is a decreasing continuous function and by Lyapunov theorem for Differential inclusions [Deimling, 1992; Prop. 14.1 p. 205] (this does not require the differentiability of Lyapunov function), the origin $(e_{\theta_1}, e_{\theta_2}, e_{w1}, e_{w2}) = 0$ is reached in finite time for each initial condition. □

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