Langrangian Dynamics of Mechanical Systems

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ABSTRACT

The literature on dynamics is rather rich both in quantity and quality. Most of it, however, is written by scientists and meant for scientists, not engineers. The aftermath is that the treatment of such topics as the dynamical analysis of nonholonomic systems is not presented with due detail, in many cases. A few exceptions are the books (meant for engineers) by Meirovitch [5], Greenwood [6] and Kane [7]. The latter introduces an original approach to the analysis of nonholonomic dynamical systems. This was further elaborated by Passerello and Huston [8]. In the present paper the approaches in [7] and [8] are treated with more formalism and oriented to computer modelling.

Contrary to the usual practice of deriving dynamical equations from the "Principle of Virtual Work", which requires the definition of virtual displacements, the author derives those equations starting from the "First Law of Thermodynamics", in an attempt to rationalize the whole formulation. Moreover, the concepts of generalized force, generalized impulse and generalized momentum are formally defined, as well as the concept of nonholonomic constraint. With regard to the latter, necessary and sufficient conditions are formally expressed for a kinamatic constraint to be holonomic.

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NOMENCLATURE

$\underset{\sim}{x}$:	lowe -case underlined character. An n-dimensional column vector.
$\underset{\sim}{A}$:	upper-case underlined character. An mxn matrix
$\underset{\sim}{x}^T$, $\underset{\sim}{A}^T$:	the transpuse of $\underset{\sim}{x}$ or, correspondingly, of $\underset{\sim}{A}$
$\underset{\sim}{x}^T\underset{\sim}{y}$, $\underset{\sim}{x}\cdot\underset{\sim}{y}$:	the scalar or inner product of n-dimensional vectors $\underset{\sim}{x}$ and $\underset{\sim}{y}$.
$\underset{\sim}{x}^T\underset{\sim}{A}\underset{\sim}{y}$:	a (scalar) bilinear form of m-dimensional vector $\underset{\sim}{x}$ and n-dimensional vector $\underset{\sim}{y}$, associated with mxn matrix $\underset{\sim}{A}$
$\underset{\sim}{q}, \dot{\underset{\sim}{q}}$:	the vector of generalized coordinates or, correspondingly, of generalized velocity.
$\frac{\partial f}{\partial \underset{\sim}{x}}$:	the partial derivative of the scalar f with respect to n-dimensional vector $\underset{\sim}{x}$. It is an n-dimensional column vector whose ith component is the partial derivative of f with respect to x_i, the ith component of $\underset{\sim}{x}$.
$\frac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{x}}$:	The partial derivative of the m-dimensional vector $\underset{\sim}{y}$ with respect to the n-dimensional vector $\underset{\sim}{x}$. It is an mxn matrix whose (i,j) element is the partial derivative of y_i with respect to x_j.

LAGRANGIAN DYNAMICS OF MECHANICAL SYSTEMS

1. PRELIMINARY CONCEPTS

1.1. Fundamental definitions

Broadly speaking a system is a set of objects that interact with each other. Thus, the set of cellestial objects constituting our gallaxy is in fact a system, for they interact with each other through gravitational forces. The different cells of a living organism constitute an additional example of a system.

A system is mechanical if, in the first place, it is constituted by mechanical elements. These are: particles, rigid bodies, continua, springs and dashpots. But a set of of such elements in itself does not constitute a mechanical system, unless its elements interact with each other through forces or exchange of mass and momentum. The state of a system is a property of the system allowing one to predict the behaviour of the system in time, i.e. the changes it undergoes. This state is constituted by a set of variables which, obviously, are referred to as state variables. If these variables are grouped within a vector, one speaks of the state vector of the system. At this stage one should make the distinction between two broad classes of systems, namely those being characterised by a finite-dimensional state vector, and those by an infinitely-dimensional state vector. The latter always refer to continua, i.e. beams, plates, shells, fluids, etc., but these will not be studied here. Thus, the mechanical systems that will be dealt with are the so-called lumped parameter systems, i.e. those composed of concentrated masses, springs, dashpots and rigid bodies.

Since the reader is assumed to be familiar with the basic concepts of mechanics, it will be taken for granted that the notions of particle, rigid body, spring, dashpot, mass, force, time, velocity and acceleration need no further discussion. Nevertheless, if needed, an extensive account of such concepts can be found in [1]. Before proceeding , however, a survey of kinematics is needed.

1.2. Kinematics of mechanical systems

The basic concepts of Kinematics of rigid bodies and their couplings are extensively discussed in [2], and often reference will be made to the material contained there. Here it will be recalled that the position vector of a point and its first two derivatives, velocity and acceleration, are denoted by $\underset{\sim}{r}$, $\dot{\underset{\sim}{r}}$ or $\underset{\sim}{v}$ and $\dot{\underset{\sim}{v}}$ or $\ddot{\underset{\sim}{r}}$, respectively. These variables are three-dimensional vector functions of time. The rotation and the angular velocity (or spin) matrices of a rigid body are tensor functions of time and are denoted by $\underset{\sim}{Q}$ and $\underset{\sim}{\Omega}$, respectively. Moreover, the angular acceleration matrix of a rigid body is given by $\dot{\underset{\sim}{\Omega}} + \underset{\sim}{\Omega}^2$. The relationship between $\underset{\sim}{Q}$ and $\underset{\sim}{\Omega}$ is

$$\underset{\sim}{\Omega} = \dot{\underset{\sim}{Q}}\underset{\sim}{Q}^T \qquad (1.2.1.)$$

Theorems concerning the foregoing variables are discussed in detail in [2].

1.3. Dynamical variables

The actions of the environment on a particle are reduced to a single concentrated force that will be denoted by $\underset{\sim}{f}$, whereas those acting on a rigid body can be either a concentrated force $\underset{\sim}{f}$ acting at a given point of the body, a distributed force $\underset{\sim}{f}'$ acting on a given portion of the surface of the body, a concentrated moment with respect to a point O, denoted by

ℓ_0 or a distributed moment with respect to point 0, acting on a given portion of the surface, that will be denoted by ℓ'_0. In many instances forces and moments can be assumed to be concentrated. The mass of either a particle or a rigid body is denoted by m. Hence, the momentum of a particle is denoted by

$$\underset{\sim}{p} = m\underset{\sim}{v} \tag{1.3.1}$$

$\underset{\sim}{v}$ being the velocity of the particle. The expression given in eq. (1.3.1) represents the momentum of a rigid body if m is the mass of the body and $\underset{\sim}{v}$ is the velocity of its center of mass. The angular momentum of a particle about a given point 0 is just the moment of the momentum of that particle about that point. Thus, denoting by $\underset{\sim}{r}$ the position vector of the particle with respect to 0, the angular momentum of the particle with respect to this point is given by

$$\underset{\sim}{h}_0 = \underset{\sim}{r} \times \underset{\sim}{p} \tag{1.3.2}$$

If Ω_{jk} is a component of the angular velocity matrix of a rigid body, its corresponding angular velocity vector having the components ω_i, then the relationship between both sets of components is

$$\omega_i = \tfrac{1}{2}\epsilon_{ijk}\,\Omega_{kj} \tag{1.3.3}$$

where $\underset{\sim}{\omega}$ is referred to as the axial vector [3] of $\underset{\sim}{\Omega}$ and is also expressed as

$$\underset{\sim}{\omega} = \text{vect}\,(\underset{\sim}{\Omega}) \tag{1.3.3a}$$

Letting $\underset{\sim}{I}_0$ denote the inertia tensor of a rigid body about point 0, the angular momentum of that body about the same point is given by

$$\underset{\sim}{h}_0 = \underset{\sim}{I}_0 \cdot \underset{\sim}{\omega} \tag{1.3.4}$$

The kinetic energy of a particle of mass m moving with velocity $\underset{\sim}{v}$ is given as

$$T = \frac{1}{2} m \underset{\sim}{v}^T \underset{\sim}{v} = \frac{1}{2} m \underset{\sim}{v}^2 \quad (1.3.5)$$

The kinetic energy of a rigid body of mass m and moment of inertia $\underset{\sim}{I}_C$ with respect to its mass center, moving so that the velocity of its mass center is $\underset{\sim}{v}$ and its angular velocity vector is $\underset{\sim}{\omega}$, is given by

$$T = \frac{1}{2} m \underset{\sim}{v}^T \underset{\sim}{v} + \frac{1}{2} \underset{\sim}{\omega}^T \underset{\sim}{I}_C \underset{\sim}{\omega} \quad (1.3.6)$$

The impulse of a given concentrated force $\underset{\sim}{f}$ during the time interval (t_1, t_2), acting on a given particle, is given by

$$\underset{\sim}{a} = \int_{t_1}^{t_2} \underset{\sim}{f}(t)\, dt \quad (1.3.7)$$

The angular impulse of that force about a given point 0 is, then,

$$\underset{\sim}{b}_0 = \int_{t_1}^{t_2} \underset{\sim}{r} \times \underset{\sim}{f}(t) dt \quad (1.3.8)$$

The angular impulse of the concentrated moment $\underset{\sim}{\ell}_0$ about a point 0, is given by

$$\underset{\sim}{b}_0 = \int_{t_1}^{t_2} \underset{\sim}{\ell}_0(t) dt \quad (1.3.9)$$

The power developed by a force $\underset{\sim}{f}$ acting on a particle moving with velocity $\underset{\sim}{v}$ is given by

$$\mathcal{P} = \underset{\sim}{f}^T \underset{\sim}{v} \quad (1.3.10)$$

Should that force act at a given point P of a rigid body, which moves with velocity $\underset{\sim}{v}$, the same expression would be valid for the power developed by that force. The power developed by a moment about a point 0, ℓ_0, acting on a body moving with angular velocity ω, is

$$U = \ell_0^T \omega \tag{1.3.11}$$

If the force f is applied at the mass center C of a rigid body moving with velocity v and simultaneously the moment about C, ℓ_C, is applied to the rigid body, the total power developed by the force and the moment is

$$U = f^T v + \ell_C^T \omega \tag{1.3.12}$$

1.4 Dynamical variables for mechanical systems

Now the foregoing concepts are defined for a mechanical system composed of several particles and rigid bodies. Assume the system consists of p particles and r rigid bodies. Moreover, assume that a force f_i acts on the i *th* particle and a force f_j acts at the mass center of the j*th* rigid body. The resultant force f acting on the system is just

$$f = \sum_{i=1}^{p} f_i + \sum_{j=1}^{r} f_j \tag{1.4.1}$$

Should the moment ℓ_{0j} about the same point 0 act on the rigid body, the resultant moment ℓ_0 about point 0 acting on the system would then be

$$\ell_0 = \sum_{j=1}^{r} \ell_{0j} \tag{1.4.2}$$

The kinetic energy of the entire system is, correspondingly,

$$T = \sum_{i=1}^{p} \frac{1}{2} m_i v_i^T v_i + \sum_{j=1}^{r} \frac{1}{2} m_j v_j^T v_j + \sum_{j=1}^{r} \frac{1}{2} \omega_j^T I_{Cj} \omega_j \tag{1.4.3}$$

where m_i and m_j are the mass of the *ith* particle and that of the *jth* rigid body, respectively. Correspondingly v_i and v_j represent the velocity of the *ith* particle and that of the mass center of the *jth* body. Analogously, I_{Cj} and ω_j represent the moment of inertia of the *jth* rigid body about its mass center and its angular velocity, respectively.

If f_i and f_j represent the forces acting on the *ith* particle and at the mass center of the *jth* rigid body, respectively, and ℓ_{Cj} the moment about the mass center of the *jth* body acting on this body moving with angular velocity ω_j, then the power developed on the entire system is given by

$$U = \sum_{i=1}^{p} f_i^T v_i + \sum_{j=1}^{n} f_j^T v_j + \sum_{j=1}^{n} \ell_{Cj}^T \omega_j \qquad (1.4.4)$$

1.5. Generalized variables

In dealing with mechanical systems consisting of either one single mass or one single rigid body, it may suffice to describe their states with the aid of the 3-dimensional position vector, r, the rotation matrix Q and their time derivatives. Furthermore, these variables can be suitably described by introducing either Cartesian coordinates or any other type like cylindrical or spherical. However, in dealing with systems comprising several masses and rigid bodies, the description of their states through the aforementioned coordinates becomes so cumbersome that the arising equations turn out to be practically unhandable. Hence a different type of description must be resorted to. Consider, for instance, the manipulator depicted in Fig. 1.5.1, composed of the three "links" AB, BC and CD, carrying the "hand" D, that, for the present purposes, can be thought of as being welded, i.e. rigidly attached to the link CD. If links AB, BC and CD are assumed to have lengths ℓ_1 , ℓ_2 and ℓ_3 and the Cartesian coordinates x, y of each point A, B, C and D are subindexed with the corresponding letter, the coordinates of D can be expressed as

$$x_D = x_B + x_C + \ell_3\cos(\theta_1 + \theta_2 + \theta_3)$$

$$y_D = y_B + y_C + \ell_3\sin(\theta_1 + \theta_2 + \theta_3)$$

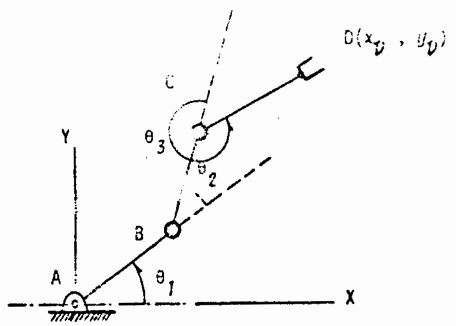

Fig 1.5.1 Three-link planar manipulator

This would require the use of seven variables, namely x_B, y_B, x_C, y_C, θ_1 , θ_2 and θ_3. These variables, however, are not independent, for they observe the relationships

$$x_B = \ell_1\cos\theta_1 \ , \ y_B = \ell_1\sin\theta_1$$

$$x_C = \ell_1\cos\theta_1 + \ell_2\cos(\theta_1 + \theta_2), y_C = \ell_1\sin\theta_1 + \ell_2\sin(\theta_1 + \theta_2)$$

$$x_D = \ell_1\cos\theta_1 + \ell_2\cos(\theta_1 + \theta_2) + \ell_3\cos(\theta_1 + \theta_2 + \theta_3)$$

$$y_D = \ell_1\sin\theta \ + \ell_2\sin(\theta_1 + \theta_2) + \ell_3\sin(\theta_1 + \theta_2 + \theta_3)$$

Hence, the position (and consequently any derivative of this) of points B, C and D can be completely described by the three variables θ_1, θ_2 and

θ_3. These, however, are not necessarily related to any particular set of polar coordinates. Such a set of variables, pertaining to the overall system, is referred to as the generalized coordinates of the system. Cartesian coordinates of some particular points are not prevented from being regarded as generalized coordinates of a system, however. In fact, one could have also taken as generalized coordinates of the system shown in Fig. 1.5.1 the following: x_B, x_C and x_D, the remaining ones being related to these by the corresponding geometric relationships that, nevertheless, would be cumbersome to deal with. Generalized coordinates are thus any set of variables that completely describe the geometrical configuration of the system under study. These variables are, then, either distances or angles supplied with signs about a reference. These variables can be orderly grouped within an array that is referred to as the vector of generalized coordinates and is henceforth represented with $\underset{\sim}{q}$. For the example of Fig. 1.5.1 one then has

$$\underset{\sim}{q} = (\theta_1, \theta_2, \theta_3)^T$$

Thus, the position vector of the *ith* particle of a mechanical system or of the mass center of the *ith* rigid body of the system is a function of the generalized coordinates. One then has

$$\underset{\sim}{r}_i = \underset{\sim}{r}_i(\underset{\sim}{q})$$

The time derivative of $\underset{\sim}{q}$ is obviously referred to as the vector of generalized velocities and is represented as $\dot{\underset{\sim}{q}}$. Vector $\underset{\sim}{q}$ being a function of time, then makes each $\underset{\sim}{r}_i$ a function of time, in turn, though through $\underset{\sim}{q}$. The velocity of either the *ith* particle or of the center of mass of the *ith* rigid body of a mechanical system, then can be expressed as

$$\underset{\sim}{v}_i \equiv \dot{\underset{\sim}{r}}_i = \frac{\partial \underset{\sim}{r}_i}{\partial \underset{\sim}{q}} \dot{\underset{\sim}{q}} + \frac{\partial \underset{\sim}{r}_i}{\partial t} \tag{1.5.1}$$

From the foregoing relationship one readily obtains the following interesting result

$$\frac{\partial \dot{\underset{\sim}{r}}_i}{\partial \dot{\underset{\sim}{q}}} = \frac{\partial \underset{\sim}{r}_i}{\partial \underset{\sim}{q}} \tag{1.5.2}$$

which is a relationship that will be very often resorted to.

Given the rotation matrix of body i of a mechanical system, this can also be regarded as a function of $\underset{\sim}{q}$. In fact, if the Eulerian angles are taken as the generalized variables of a rigid body, taking the body from configuration 0 to configuration 3, as depicted in Fig. 1.5.2, where multually orthogonal axes X, Y, Z have been attached to the body, one has [1,p.19][1.1].

$$\underset{\sim}{Q} = \begin{bmatrix} c\theta c\phi c\psi - s\phi s\psi & -c\theta c\phi s\psi - s\phi c\psi & s\theta c\phi \\ c\theta s\phi c\psi + c\phi s\psi & -c\theta s\phi s\psi + c\phi c\psi & s\theta s\phi \\ -s\theta c\psi & s\theta s\psi & c\theta \end{bmatrix}$$

where $\underset{\sim}{Q}$ has not been indexed for simplicity.

Thus,

$$\dot{\underset{\sim}{Q}} = \frac{\partial \underset{\sim}{Q}}{\partial \underset{\sim}{q}} \dot{q} + \frac{\partial \underset{\sim}{Q}}{\partial t} \tag{1.5.3}$$

where $\partial \underset{\sim}{Q}/\partial \underset{\sim}{q}$ is a triadic, i.e. a third-rank tensor, whose components depend upon three indices, as is shown next

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1.1 $c\,x \equiv \cos x$, $s\,x \equiv \sin x$, henceforth

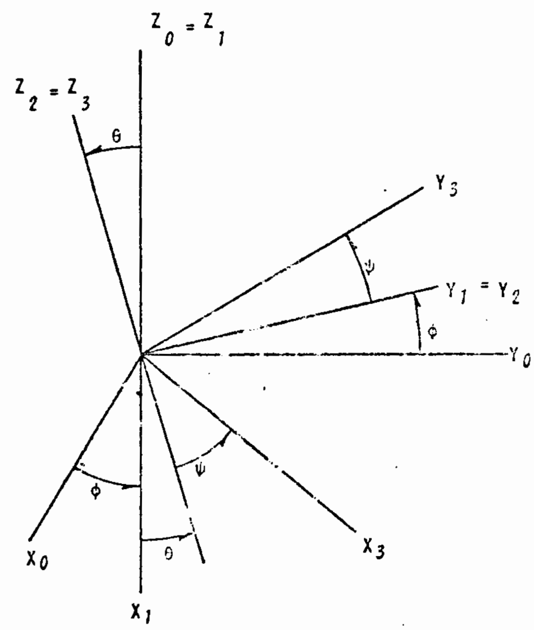


Fig. 1.5.2 Euler's angles for a igid-body rotation about a point O.

$$(\dot{Q})_{ij} = \frac{\partial Q_{ij}}{\partial q_k}\dot{q}_k + \frac{\partial Q_{ij}}{\partial t} \qquad (1.5.4)$$

i.e. the (i,j,k) component of $\partial Q/\partial q$ is $\partial Q_{ij}/\partial q_k$

The angular velocity matrix, Ω, of the rigid body under study, then has the following (i,j) component

$$\Omega_{ij} = \dot{Q}_{i\ell}Q_{j\ell} = \frac{\partial Q_{i\ell}}{\partial q_k}\dot{q}_k Q_{j\ell} + \frac{\partial Q_{i\ell}}{\partial t} Q_{j\ell} = \frac{\partial Q_{i\ell}}{\partial q_k} Q_{j\ell}\dot{q}_k + \frac{\partial Q_{i\ell}}{\partial t} Q_{j\ell} \qquad (1.5.4a)$$

which, in compact form, can be written as

$$\Omega = \frac{\partial Q}{\partial q} Q^T\dot{q} + \frac{\partial Q}{\partial t} Q^T \qquad (1.5.5)$$

and the relationship

$$\frac{\partial \Omega}{\partial \dot{q}} = \frac{\partial Q}{\partial q} Q^T \qquad (1.5.6)$$

readily follows. Of course, the angular velocity Ω of the *ith* rigid body is

$$\Omega_i = \frac{\partial Q_i}{\partial q} Q_i^T\dot{q} + \frac{\partial Q_i}{\partial t} Q_i^T \qquad (1.5.7)$$

or, making use of eq. (1.5.6),

$$\Omega_i = \frac{\partial \Omega_i}{\partial \dot{q}}\dot{q} + \frac{\partial Q_i}{\partial t} Q_i^T \qquad (1.5.8)$$

Analogously, introducing eq. (1.5.2) into eq (1.5.1), one obtains

$$v_i = \frac{\partial r_i}{\partial q}\dot{q} + \frac{\partial r_i}{\partial t} = \frac{\partial v_i}{\partial \dot{q}}\dot{q} + \mu_i \qquad (1.5.9)$$

By taking the axial vector of both sides of eq (1.5.8), one has[1.2]

$$\text{vect}(\Omega_i) = \text{vect}\left(\frac{\partial \Omega_i}{\partial \dot{q}}\dot{q}\right) + \text{vect}\left(\frac{\partial Q_i}{\partial t} Q_i^T\right) = \left[\frac{\partial}{\partial \dot{q}}\text{vect}(\Omega_i)\right]\dot{q} + \text{vect}\left(\frac{\partial Q_i}{\partial t} Q_i^T\right)$$

1.2 These results follow readily from definition (1.3.3) for the vector of a matrix and eq (1.5.4a) and, recalling definition (1.3.3), the relationship

$$\underset{\sim}{\omega}_i = \frac{\partial \underset{\sim}{\omega}_i}{\partial \dot{\underset{\sim}{q}}} \dot{\underset{\sim}{q}} + \text{vect} \left(\frac{\partial Q_i}{\partial t} Q_i^T \right) \qquad (1.5.10)$$

follows immediately.

Now, introducing vector $\underset{\sim}{\nu}_i \equiv \underset{\sim}{\nu}_i(\underset{\sim}{q},t)$ defined as

$$\underset{\sim}{\nu}_i = \text{vect} \left(\frac{\partial \underset{\sim}{Q}_i}{\partial t} \underset{\sim}{Q}_i^T \right)$$

eq. (1.5.10) can be rewritten as

$$\underset{\sim}{\omega}_i = \frac{\partial \underset{\sim}{\omega}_i}{\partial \dot{\underset{\sim}{q}}} \dot{\underset{\sim}{q}} + \underset{\sim}{\nu}_i \qquad (1.5.10a)$$

Moreover, since $\underset{\sim}{\omega}_i = \underset{\sim}{\omega}_i(\underset{\sim}{q},\dot{\underset{\sim}{q}},t)$, one has

$$\dot{\underset{\sim}{\omega}}_i = \frac{\partial \underset{\sim}{\omega}_i}{\partial \underset{\sim}{q}} \dot{\underset{\sim}{q}} + \frac{\partial \underset{\sim}{\omega}_i}{\partial \dot{\underset{\sim}{q}}} \ddot{\underset{\sim}{q}} + \frac{\partial \underset{\sim}{\omega}_i}{\partial t} \qquad (1.5.10b)$$

Hence, the following relationship

$$\frac{\partial \dot{\underset{\sim}{\omega}}_i}{\partial \ddot{\underset{\sim}{q}}} = \frac{\partial \underset{\sim}{\omega}_i}{\partial \dot{\underset{\sim}{q}}} \qquad (1.5.10c)$$

Next, both relationships (1.5.9) and (1.5.10a) are substituted into eq. (1.4.4), thus obtaining the power supplied to a mechanical system by its environment, as

$$U = \left(\sum_{i=1}^{p} \underset{\sim}{f}_i^T \frac{\partial \underset{\sim}{v}_i}{\partial \dot{\underset{\sim}{q}}} + \sum_{j=1}^{r} \underset{\sim}{f}_j^T \frac{\partial \underset{\sim}{v}_j}{\partial \dot{\underset{\sim}{q}}} + \sum_{j=1}^{r} \ell_{Cj}^T \frac{\partial \underset{\sim}{\omega}_j}{\partial \dot{\underset{\sim}{q}}} \right) \cdot \dot{\underset{\sim}{q}} + $$

$$+ \sum_{i=1}^{p} \underset{\sim}{f}_i^T \underset{\sim}{\mu}_i + \sum_{j=1}^{r} \underset{\sim}{f}_j^T \underset{\sim}{\mu}_j + \sum_{j=1}^{r} \underset{\sim}{\ell}_{Cj}^T \underset{\sim}{\nu}_j \qquad (1.5.11)$$

which is an expression of the form

$$U = \underset{\sim}{\phi}^T \dot{\underset{\sim}{q}} + \psi \tag{1.5.11a}$$

The first term of the right-hand side of expression (1.5.11a) is the inner product of a quantity $\underset{\sim}{\phi}$ times a generalized velocity, its units being those of power. In elementary mechanics, to obtain power, velocity must be multiplied by force. Hence, by similitude, $\underset{\sim}{\phi}$ is defined as the generalized force acting on the system, also referred to as the active generalized force or the external generalized force, as opposed to the inertial generalized force, yet to be defined. The remaining term, ψ, will be seen to play no relevant role in the present formulation and hence, will not be further discussed for the time being. One then has defined

$$\underset{\sim}{\phi} = \sum_{i=1}^{p+r} \left(\frac{\partial \underset{\sim}{v}_i}{\partial \dot{\underset{\sim}{q}}}\right)^T \underset{\sim}{f}_i + \sum_{j=1}^{r} \left(\frac{\partial \underset{\sim}{\omega}_j}{\partial \dot{\underset{\sim}{q}}}\right)^T \underset{\sim}{\ell}_{Cj} \tag{1.5.12}$$

which is an n-dimensional vector, its kth component being[1.3]

$$\phi_k = \sum_{i=1}^{p+r} \frac{\partial v_m^i}{\partial \dot{q}_k} f_m^i + \sum_{j=1}^{r} \frac{\partial \omega_m^j}{\partial \dot{q}_k} \ell_{Cm}^j \tag{1.5.12a}$$

Now, if impulsive actions take place on the particles and rigid bodies of a mechanical system, a generalized impulse can be defined analogously. Indeed, assume $\underset{\sim}{a}_i$ is the impulse acting on the ith particle of such a system and $\underset{\sim}{a}_j$ that on the center of mass of the jth rigid body. Moreover, let $\underset{\sim}{b}_{Cj}$ be the angular impulse with respect to the mass center of the jth rigid body, acting on this body. The generalized impulse acting on the system is then defined as:

[1.3] The repeated index m, ranging from 1 to 3, implies sum over it.

$$\underset{\sim}{\theta} = \sum_{i=1}^{p+r} \left(\frac{\partial \underset{\sim}{v}}{\partial \dot{\underset{\sim}{q}}}\right)^T \underset{\sim}{a}_i + \sum_{j=1}^{r} \left(\frac{\partial \underset{\sim}{\omega}_j}{\partial \dot{\underset{\sim}{q}}}\right)^T \underset{\sim}{b}_{Cj} \tag{1.5.13}$$

by analogy with definition (1.5.12).

Denoting by $\dot{\underset{\sim}{v}}_i$ the acceleration of the *ith* particle or that of the mass center of the *ith* rigid body of a mechanical system, and by m_i the mass of the *ith* particle or, correspondingly, that of the *ith* rigid body, the *ith* inertia force is defined in elementary mechanics as

$$\underset{\sim}{f}_i^* \equiv - m_i \dot{\underset{\sim}{v}}_i \tag{1.5.14}$$

The inertia cuple of the *ith* rigid body of the mechanical system under study, with respect to its mass center, is correspondingly defined as

$$\underset{\sim}{\ell}_{Ci}^* \equiv - \underset{\sim}{I}_{Ci}\dot{\underset{\sim}{\omega}}_i - \underset{\sim}{\omega}_i \times \underset{\sim}{I}_{Ci}\underset{\sim}{\omega}_i \tag{1.5.15}$$

The inertia generalized force of the system under consideration is defined as

$$\underset{\sim}{\phi}^* \equiv - \sum_{i=1}^{p+r} \left(\frac{\partial \underset{\sim}{v}_i}{\partial \dot{\underset{\sim}{q}}}\right)^T m_i \dot{\underset{\sim}{v}}_i - \sum_{j=1}^{r} \left(\frac{\partial \underset{\sim}{\omega}_j}{\partial \dot{\underset{\sim}{q}}}\right)^T (\underset{\sim}{I}_{Cj}\dot{\underset{\sim}{\omega}}_j + \underset{\sim}{\omega}_j \times \underset{\sim}{I}_{Cj}\underset{\sim}{\omega}_j) \tag{1.5.16}$$

by analogy with definition (1.5.12).

The generalized momentum is in turn defined as

$$\underset{\sim}{\sigma}^* \equiv - \sum_{i=1}^{p+r} \left(\frac{\partial \underset{\sim}{v}_i}{\partial \dot{\underset{\sim}{q}}}\right)^T m_i \underset{\sim}{v}_i - \sum_{j=1}^{r} \left(\frac{\partial \underset{\sim}{\omega}_j}{\partial \dot{\underset{\sim}{q}}}\right)^T \underset{\sim}{I}_{Cj}\underset{\sim}{\omega}_j \tag{1.5.17}$$

again, by analogy with definition (1.5.12).

The degree of fredom of a mechanical system equals the necessary and sufficient number of generalized coordinates that describe uniquely the geometric configuration of the system. Hence, it can also be defined as the largest possible number of *independent* generalized coordinates of the system. For example, the degree of freedom of the system shown in Fig 1.5.1 is three.

From eqs (1.3.10-12) it is clear that the power developed by a force acting on a particle, or at the mass center of a rigid body, vanishes either if the force is perpendicular to the velocity of the point on which it acts, or if the velocity of this point vanishes. Analogously, the power developed by a torque acting on a rigid body vanishes either if the torque is perpendicular to the angular velocity or if the latter vanishes. This is a well-known result from elementary mechanics. It is worth mentioning it here, however, because it makes clear that the power contibution of such forces and torques on the mechanical system vanishes and hence, does not appear in either sum of eq (1.4.4). As a consequence, then, at light of the derivation of this Section, the contributions to the overall generalized active force, of forces and torques acting upon a mechanical system, that develop zero power on particular particles or rigid bodies of the system, vanish as well. Hence they need not be counted for in computing the generalized active force of the mechanical system.

Similar arguments lead to analogous results for generalized inertia forces, impulses and momenta.

The foregoing results are advantageous in constructing the mathematical models of mechanical dynamical systems for, under these, reaction or contact forces on rolling bodies or on frictionless surfaces are eliminated *ipso facto* as unknowns, which thus simplifies the formulation of the model. They present equally a disadvantage, however. In fact, if the said reaction or contact forces are needed for design purposes, i.e. in determining work loads on mechanical elements, one cannot obtain them directly using the Lagrangian formulation presented in the following sections. These forces and moments can be computed by relaxing the constraints of interest and introducing them at the final stage of the model formulation. Such a technique appears in [7,pp. 207-210].

1.6. Newton's equations of motion

Newton's laws of motion for a given particle of mass m, acted upon by a force $\underset{\sim}{f}$ are expressed as

$$\underset{\sim}{f} = m\ddot{\underset{\sim}{r}} \tag{1.6.1}$$

$\ddot{\underset{\sim}{r}}$ being the acceleration of the particle.

The same laws referred to a given rigid body of mass m and moment of inertia $\underset{\sim}{I}_C$ about its mass center, acted upon by a force $\underset{\sim}{f}$ applied at its mass center, C, and a moment $\underset{\sim}{\ell}_C$ with respect to C, are summarized in the following equations:

$$\underset{\sim}{f} = m\ddot{\underset{\sim}{r}} \tag{1.6.2a}$$

$$\underset{\sim}{\ell}_C = \underset{\sim}{I}_C \cdot \dot{\underset{\sim}{\omega}} + \omega \times \underset{\sim}{I}_C \cdot \underset{\sim}{\omega} \tag{1.6.2b}$$

Newton's laws are now applied to a system composed of p particles and r rigid bodies: If force $\underset{\sim}{f}_i$ acts upon the *ith* particle, force $\underset{\sim}{f}_j$ at the mass center of the *jth* rigid body and moment $\underset{\sim}{\ell}_{Cj}$ is the moment with respect to the center of mass of the *jth* rigid body, acting upon that body, one then has

$$\underset{\sim}{f}_i = \underset{\sim}{m}_i \ddot{\underset{\sim}{r}}_i \quad , \; i = 1,\ldots,p \tag{1.6.3a}$$

$$\underset{\sim}{f}_j = \underset{\sim}{m}_j \ddot{\underset{\sim}{r}}_j \quad , \; j = 1,\ldots,r \tag{1.6.3b}$$

$$\underset{\sim}{\ell}_{Cj} = \underset{\sim}{I}_{Cj} \dot{\omega}_j + \omega_j \times \underset{\sim}{I}_{Cj} \cdot \omega_j \; , \; j = 1,\ldots,r \tag{1.6.3c}$$

By addition of the p equations (1.6.3a) and of the r equations (1.6.3b) and then by addition of the two resulting sums, one has

$$\sum_{i=1}^{p+r} \underset{\sim}{f}_i = \sum_{i=1}^{p+r} m_i \ddot{\underset{\sim}{r}}_i \tag{1.6.4a}$$

Next, addition of the n equations (1.6.3b) leads to

$$\sum_{j=1}^{n} \ell_{Cj} = \sum_{j=1}^{n} (\underset{\sim}{I}_{Cj} \cdot \dot{\underset{\sim}{\omega}}_j + \underset{\sim}{\omega}_j \times \underset{\sim}{I}_{Cj} \cdot \underset{\sim}{\omega}_j) \tag{1.6.4b}$$

Eqs (1.6.4a and b) are expressions of Newton's laws to be applied to systems of particles and rigid bodies.

2. DYNAMICAL EQUATIONS OF MECHANICAL SYSTEMS

2.1. The First Law of Thermodynamics

The First Law of Thermodynamics, as such, is the result of observation and hence, cannot be derived, as neither can Newton's laws of motion. This law states that the energy of a physical system remains constant throughout the time. Within this context, the physical system under study will be considered to be composed of the mechanical system at hand plus its environment. Thus, the First Law of Thermodynamics can be rephrased as: "The power supplied to a mechanical system by its environment equals the time rate of change of the energy of the system".

The time rate of change of the energy of the system is also referred to as its internal power or else as to the power developed by the system, as opposed to the power supplied by the environment to the system, an expression for which was obtained as eq. (1.5.11). Now, the energy of the system is also called its internal energy and equals the time integral of its internal power within a certain time interval. The internal energy of a mechanical system can be of three different types: kinetic, potential and nonrecoverable. The kinetic energy of a mechanical system was already defined in Sect. 1.4, an expression for which was obtained as eq. (1.4.3). The potential energy is next discussed.

The action of the environment on a mechanical system takes on several forms, but in dealing with finite-degree-of-freedom purely mechanical systems, this can be only of two different types, namely forces and moments. These can in turn be supplied by either natural or man-made means, such as motors, springs, etc. In any case, forces and moments can arise from vector fields arising in turn from scalar fields. A scalar (vector) field is a scalar (vector) function whose domain is defined as a portion of the physical space, i.e., whose argument is the position vector $\underset{\sim}{r}$ of a point P of a given region of space. Examples of scalar fields are the atmospheric pressure and temperature. Examples of vector fields are the force of gravity and the velocity of the particles of water contained in a waterfall. A scalar field f and a vector field $\underset{\sim}{f}$ can be expressed thus, as

$$f = f(\underset{\sim}{r}),\ \underset{\sim}{f} = \underset{\sim}{f}(\underset{\sim}{r}) \qquad (2.1.1a)$$

respectively. If these fields change in time, then they should be expressed as

$$f = f(\underset{\sim}{r},t),\ \underset{\sim}{f} = \underset{\sim}{f}\ (\underset{\sim}{r},t) \qquad (2.1.1b)$$

respectively.

Assuming that the vector field $\underset{\sim}{f}(\underset{\sim}{r})$ and the scalar field $V(\underset{\sim}{r})$ are related by

$$\underset{\sim}{f}\ (\underset{\sim}{r}) = -\frac{\partial V}{\partial \underset{\sim}{r}} \qquad (2.1.2)$$

i.e. assuming that $\underset{\sim}{f}$ is the gradient of-V, V is said to be a <u>pontential</u> of $\underset{\sim}{f}$. Notice that $\underset{\sim}{f}$ can have several potentials, all of them differing by a function not containing $\underset{\sim}{r}$ explicitly or else, if time does not play any role, by a constant. If the force $\underset{\sim}{f}$ defined in eq. (2.1.2) acts upon a particle of mass m during the time interval $(t_0,\ t_1)$, the energy supplied by this force during the said time interval to the particle, assumed to move with velocity $\underset{\sim}{v}$, can be computed readily as the time integral of the power $\underset{\sim}{f}.\underset{\sim}{v}$, i.e.

$$E = \int_{t_0}^{t_1} \underset{\sim}{f}\cdot\underset{\sim}{v}\ dt \qquad (2.1.3)$$

Substitution of eq. (2.1.2) into eq. (2.1.3) and of $\underset{\sim}{v}$ by $d\underset{\sim}{r}/dt$, eq. (2.1.3) becomes

$$E = -\int_{t_0}^{t_1} \frac{\partial V}{\partial \underset{\sim}{r}} \cdot \frac{d\underset{\sim}{r}}{dt}\, dt = -\int_{r_0}^{r_1} \frac{\partial V}{\partial \underset{\sim}{r}}\, d\underset{\sim}{r} = -\int_{V_0}^{V_1} dV$$

Thus,

$$E = V_0 - V_1 \tag{2.1.4}$$

i.e. the energy supplied to the particle during the time interval (t_0, t_1) depends only upon its initial and final positions. This type of energy is called the <u>potential energy of the particle</u>. It should be pointed out that the potential energy of a particle at a given time always depends upon a reference position

Now, denote with $\underset{\sim}{\phi}_f$ and $\underset{\sim}{\phi}_m$ the first and the second summations appearing in eq. (1.5.12). The generalized force $\underset{\sim}{\phi}$ has thus been decomposed as

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}_f + \underset{\sim}{\phi}_m \tag{2.1.5}$$

where

$$\underset{\sim}{\phi}_f = \sum_{i=1}^{p+r} \left(\frac{\partial \underset{\sim}{r}_i}{\partial \underset{\sim}{q}}\right)^T \underset{\sim}{f}_i \tag{2.1.6a}$$

and

$$\underset{\sim}{\phi}_m = \sum_{i=1}^{r} \left(\frac{\partial \underset{\sim}{\omega}_i}{\partial \underset{\sim}{q}}\right)^T \underset{\sim}{\ell}_{ci} \tag{2.1.6b}$$

where the identity appearing in (1.5.2) was introduced into eq. (2.1.6a). Let all forces $\underset{\sim}{f}_i$ appearing in eq. (2.1.6a) arise from a given potential V_f, i.e. let

$$\underset{\sim}{f}_i = -\frac{\partial V_f}{\partial \underset{\sim}{r}_i} \tag{2.1.7}$$

Substitution of eq. (1.6.7) into eq. (2.1.6a) leads to

$$\underset{\sim}{\phi}_f = -\sum_{i=1}^{p+r} \left(\frac{\partial \underset{\sim}{r}_i}{\partial \underset{\sim}{q}}\right)^T \frac{\partial V_f}{\partial \underset{\sim}{r}_i} = -\frac{\partial V_f}{\partial \underset{\sim}{q}} \tag{2.1.8a}$$

where the second equation follows directly from the "chain rule". Similarly, if all couples $\underset{\sim}{\ell}_{Ci}$ appearing in eq. (2.1.6b), arise from a given potential V_m, it is possible[2.1] to rewrite it in the form

$$\underset{\sim}{\phi}_m = -\frac{\partial V_m}{\partial \underset{\sim}{q}} \tag{2.1.8b}$$

If now the potential V is defined as the sum of V_f and V_m, one has

$$\underset{\sim}{\phi} = -\frac{\partial V}{\partial \underset{\sim}{q}} \tag{2.1.9}$$

It may happen that only some of the forces and couples acting on a mechanical system arise from a potential. In this case, the generalized force $\underset{\sim}{\phi}$ can be decomposed as the sum of $\underset{\sim}{\phi}_p$ and $\underset{\sim}{\phi}_{np}$, $\underset{\sim}{\phi}_p$ being that part of $\underset{\sim}{\phi}$ arising from a potential, whereas $\underset{\sim}{\phi}_{np}$ that not arising from any potential. Forces and couples arising from potentials are also called "lamellar". If lamellar fields are independent of time, they are called "conservative". Those forces and couples not arising from any potential are called nonconservative. The generalized force then can be written as

$$\underset{\sim}{\phi} = -\frac{\partial V}{\partial \underset{\sim}{q}} + \underset{\sim}{\phi}_{np} \tag{2.1.10}$$

Thus far, kinetic and potential forms of mechanical energy have been discussed. Nonrecoverable form of energy is that being dissipated as heat, due to either friction or viscosity. This nonrecoverable form of energy is

[2.1] The details involved are omitted for they are unnecessary to pursue the discussion.

The power developed by dry-friction forces is thus

$$U_d = -F_o v \operatorname{sgn}(v)$$

where $v \equiv v_2 - v_1$ and sgn (v) is the signum function, which equals+1 or -1, depending upon whether v is positive or negative. If v = 0, clearly there is no power developed and hence sgn(0) = 0. The power developed by linear viscous forces (the only ones to be considered here unless otherwise indicated) is then

$$U_v = -cv^2$$

The work developed by each of these forces within the interval $t_1 < t < t_2$ is simply the integral of the corresponding expression, i.e.

$$W_d = -F_o \int_{t_1}^{t_2} v \operatorname{sgn}(v)\, dt$$

$$W_v = -c \int_{t_1}^{t_2} v^2\, dt$$

Linear viscous forces can be regarded as arising from a scalar function, referred to as a "dissipation function", D, if this is defined as

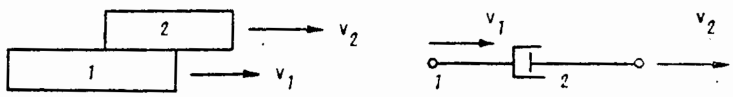
$$D = \frac{1}{2} cv^2$$

and hence the corresponding viscous force is related to D as

$$F = -\frac{\partial D}{\partial v}$$

D playing thus a role similar to a potential. Since this type of forces is nonconservative, D cannot be thought of being actually a potential. Moreover, F is not the gradient of D, since it is not obtained by differentiation of D with respect to the position vector, but with respect to the velocity.

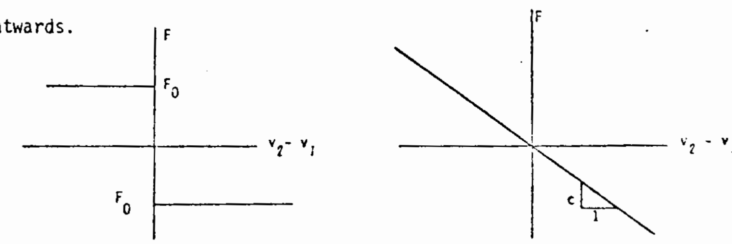
The extension of the concept of dissipation function to mechanical systems is straightforward. If only linear viscous forces are present within commonly referred to as "mechanical losses". In this context, only two forms of nonrecoverable mechanical energy will be considered: those due to dry friction and those due to viscosity. The former follow the so-called "laws of friction" and thus, are independent of velocity. Dry-friction forces acting on bodies at relative rest differ from those acting on bodies under relative motion, however. Thus, a distinction is made between static and kinetic dry-friction forces. Friction forces due to viscosity are a consequence of the internal viscosity of fluids, which can be either proportional to velocity or more complicated functions of it. Dry friction is represented by two surfaces in contact, either under relative motion or tending to have it, as shown in Fig. 2.1 (a). Viscous friction is represented by a dashpot whose end points move with different velocities, v_1 and v_2, as shown in Fig. 2.1 (b).



(a) Dry friction (b) Viscous friction

Fig.2.1.1. Friction forces

Fig.2.2 shows the friction force that body 1 exerts on body 2, shown in Fig.2.1, where the positive direction has been assumed to be that pointing rightwards.



(a) Dry-friction force (b) Linear viscous force

Fig. 2.1.2 Dry-friction and linear viscous force the system, and these are grouped within the generalized force $\underset{\sim}{\phi}_v$, the quadratic dissipation function D, defined as

$$D = \frac{1}{2} \dot{\underset{\sim}{q}}^T \underset{\sim}{C} \dot{\underset{\sim}{q}}$$

allows the computation of $\underset{\sim}{\phi}_v$ as

$$\underset{\sim}{\phi}_v = - \frac{\partial D}{\partial \dot{\underset{\sim}{q}}}$$

an expression similar to eq. (2.1.9) for lamellar fields.

2.2. Classification of mechanical systems according to their constraints

Mechanical systems, to be useful, must be composed of coupled particles and rigid bodies. The action of coupling has the effect of diminishing the degree of freedom of each element constituting the system. Coupling, thus, means introducing constraints into the system. These constraints can be readily expressed in mathematical form by introducing expressions relating the velocity of the particle or of the mass centre of a rigid body, or the angular velocity of the rigid body, to those of the remaining elements, thus limiting the range of these variables to a smaller set. These constraints, relating velocities, are then relations amongst generalized velocities, i.e. amongst derivatives of the generalized coordinates. From kinematics, it follows that these relations take on the forms of linear combinations of the generalized coordinates. Thus, they take on the form

$$\underline{A}\dot{\underline{q}} = \underline{b} \qquad (2.2.1)$$

where $\underline{A}$ is a $m \times n$ $(m<n)$ matrix, n being the number of generalized coordinates. Eq. (2.2.1) thus represents a set of m equations of the form

$$a_{i1}\dot{q}_1 + a_{i2}\dot{q}_2 + \ldots + a_{in}\dot{q}_n = b_i\,, \quad i=1,\ldots,m \qquad (2.2.2)$$

which can be rewritten in the form

$$\underline{a}_i^T\dot{\underline{q}} = b_i\,, \quad i=1,\ldots,m \qquad (2.2.2a)$$

If vector $\underline{a}_i$ happens to be the gradient of a given function f_i, i.e. if

$$\underline{a}_i = \frac{\partial f_i}{\partial \underline{q}} \qquad (2.2.3)$$

and if b_i happens to be the negative of the time derivative of the given function f_i, one can rewrite eq. (2.2.2a) as

$$\left(\frac{\partial f_i}{\partial \underset{\sim}{q}}\right)^T \dot{\underset{\sim}{q}} + \frac{\partial f_i}{\partial t} = \underset{\sim}{0}, \quad i = 1,\ldots,m \tag{2.2.4}$$

whose left-hand side is readily identified as the total derivative of f_i with respect to time. Thus, one can write it as

$$\dot{f}_i = 0, \quad i = 1,\ldots,m \tag{2.2.5}$$

which leads readily to

$$f_i = c_i \tag{2.2.6}$$

c_i being a constant. Thus eq. (2.2.2) turns to be integrable under the introduced assumptions. Such a constraint, possessing an integral, is referred to as a holonomic constraint[2.2]. Otherwise, such a constraint is referred to as a nonholonomic constraint. Systems containing only holonomic constraints are thus referred to as holonomic systems. Assuming a system contains both holonomic and nonholonomic constraints, the set of holonomic constraints can always be integrated to render a system of, usually nonlinear, algebraic equations of the form

$$\underset{\sim}{f}(\underset{\sim}{q},t) = \underset{\sim}{0} \tag{2.2.7}$$

whose dimension will be assumed to be $h(<m)$. Eq. (2.2.7) thus defines h relations between the generalized coordinates. This means that from that equation, h generalized coordinates can be solved for in terms of the remaining $m-h$ coordinates. For nonlinear systems, this will not, in general, be possible in closed form. Well-known numerical procedures, like that of Newton-Raphson, can be applied to solve for the involved h generalized

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2.2 Greek: holos=integer

coordinates, however. Thus, finally, one can think of a nonholonomic system as one possessing only nonholonomic constraints, its holonomic constraints being assumed to have been integrated and, from this integration, h of its components already solved for.

Whether a system is holonomic or not can be readily verified. In fact, a necessary condition for a function $f_i(q,t)$ to exist, that verifies eq. (2.2.3), is that the gradient of $\underset{\sim}{a}_i$, as defined in eq. (2.2.2a), with respect to q, a $n \times n$ matrix, be symmetric. This is not sufficient, however, to render a constraint holonomic. In fact, the term b_i of eq. (2.2.2a) should be related to f_i by

$$b_i = -\frac{\partial f_i}{\partial t} \quad , \; i=1,\ldots,m \tag{2.2.8}$$

This means that the second necessary condition to render a constraint holonomic is

$$\frac{\partial b_i}{\partial \underset{\sim}{q}} = -\frac{\partial \underset{\sim}{a}_i}{\partial t} \quad , \; i=1,\ldots,m \tag{2.2.9}$$

Holonomic systems are easier to deal with than nonholonomic ones, because they allow to reduce the number of generalized variables. In fact, by solving for h of the generalized variables from the h holonomic constraints of a holonomic system, the number of generalized coordinates reduces to n-h. Moreover, these n-h coordinates are independent, their time derivatives being linearly independent. Thus, with no loss of generality, the n generalized coordinates of a holonomic system can be thought of as being independent.

A particular case of interest arises when matrix $\underset{\sim}{A}$ and vector $\underset{\sim}{b}$ of eq (2.2.1) are constant. In this case, clearly, the $n \times m$ gradient with respect to $\underset{\sim}{q}$, of every row of $\underset{\sim}{A}$, vanishes, i.e. it is symmetric. Moreover, for each row $\underset{\sim}{a}_i$, a scalar function f_i exists whose gradient with respect to $\underset{\sim}{q}$ is $\underset{\sim}{a}_i$. In fact, this is

$$f_i = \underset{\sim}{a}_i^T \underset{\sim}{q} + b_i$$

with $\underset{\sim}{a}_i$ constant. Hence, eqs (2.2.8) and (2.2.9) hold and the constraints are holonomic.

Example 2.2.1. Constraints of a differential gear train. A model of a differential gear train is shown in Fig 2.2.1. It consists of two disks A and B. Disks A have a radius a , whereas disk B has a radius b. Disks A rotate freely about independent axes that are collinear with line AA' , whereas disk B rotates about axis OB and this one rotates in turn about AA'. Since the model represents a gear train, pure rolling is assumed.

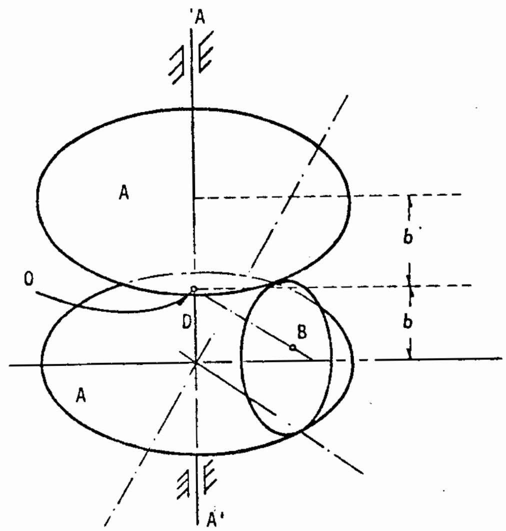

Fig 2.2.1 Kinematic model of a differential gear train

Solution: Let θ_1 and θ_2 be the angles of rotation of the upper and the lower disk A, respectively, θ_3 and θ_4 being the angles of rotation of disk B about OB and of OB about AA'. These are then the generalized coordinates of the system. Let P and Q be the contact points of disk B with the upper and the lower disk, respectively. Moreover, let $\underline{n}_1$ and $\underline{n}_2$ be unit vectors parallel to OB and to AA', directed along the positive directions of θ_3 and θ_4 (or θ_1), respectively. Finally, let $\underline{n}_3 = \underline{n}_1 \times \underline{n}_2$.

Next, denote by $\underline{v}_{PA}$ and $\underline{v}_{PB}$ the velocity of point P of disks A and B, respectively. Define $\underline{v}_{QA}$ and $\underline{v}_{QB}$ analogously. Then

$$\underline{v}_{PA} = -\dot{\theta}_1 a\, \underline{n}_3 , \quad \underline{v}_{PB} = (-\dot{\theta}_4 a + \dot{\theta}_3 b)\underline{n}_3$$

$$\underline{v}_{QA} = -\dot{\theta}_2 a\, \underline{n}_3 , \quad \underline{v}_{QB} = (-\dot{\theta}_4 a - \dot{\theta}_3 b)\underline{n}_3$$

The pure-rollling condition imposes

$$\underline{v}_{PA} = \underline{v}_{PB} \text{ and } \underline{v}_{QA} = \underline{v}_{QB}$$

Hence,

$$-\dot{\theta}_1 a = -\dot{\theta}_4 a + \dot{\theta}_3 b$$

$$-\dot{\theta}_2 a = -\dot{\theta}_4 a - \dot{\theta}_3 b$$

respectively. These are the constraint equations of the system of Fig 2.2.1, which have the form of eq (2.2.1) with

$$\dot{\underline{q}} = \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \end{pmatrix} , \quad \underline{A} = \begin{pmatrix} -a & 0 & -b & a \\ 0 & a & b & a \end{pmatrix} , \quad \underline{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since matrix $\underset{\sim}{A}$ and vector $\underset{\sim}{b}$ of these equations are constant, they are integrable, i.e. the system is holonomic. Integration of the foregoing equations yields

$$\underset{\sim}{A}\underset{\sim}{q} = \underset{\sim}{k}$$

k being a 2- dimensional constant vector. If all angles are defined 0 at t = 0, clearly $\underset{\sim}{k} = \underset{\sim}{0}$, which then yields

$$\underset{\sim}{A}\underset{\sim}{q} = \underset{\sim}{0}$$

or, in component form

$$-a\theta_1 - b\theta_3 + a\theta_4 = 0$$

$$-a\theta_2 + b\theta_3 + a\theta_4 = 0$$

The foregoing system possesses four generalized coordinates and two kinematic constraints. The system is thus holonomic, its degree of freedom being two. The latter fact means that it admits two independent inputs, which is the reason why it is essential as a transmission device in power axles of terrestrial vehicles. In fact, it allows the transmission of power to the axle AA', while allowing each of disks A to rotate at independent rates when taking a curve. To illustrate this fact, elliminate $\dot{\theta}_4$ from the velocity equations already obtained. Next, solve for $\dot{\theta}_3$ in terms of $\dot{\theta}_1$ and $\dot{\theta}_2$ from the arising equation, thus obtaining

$$\dot{\theta}_3 = \frac{a}{2b}(\dot{\theta}_2 - \dot{\theta}_1)$$

which shows clearly that the rate of rotation of disk B about its axis of symmetry is proportional to the *difference* of rates $\dot{\theta}_1$ and $\dot{\theta}_2$, which explains the name of the device. When the vehicle follows a straight course, clearly $\dot{\theta}_3 = 0$.

Example 2.2. Constraint equations on disk rolling without slipping on a horizontal surface. Consider the disk shown in Fig 2.2.2, whose movement is defined by the Cartesian coordinates (x,y,z) of its mass center, and the three orientation angles θ , ϕ and ψ, associated with the direction of the course, the tilt with respect to a vertical plane containing the tangent to the disk at the point of contact P, and the spin about the axis of symmetry of the disk. Now, define lines *1*, *2* and *3* as follows: Line *1* is the diameter of the disk parallel instantly to the tangent passing through P; line *2* is the diameter perpendicular to line *1*, and line *3* is the axis of symmetry of the disk. Next, define unit vectors $\underset{\sim}{n}_1$, $\underset{\sim}{n}_2$ and $\underset{\sim}{n}_3$ parallel to lines *1* , *2* and *3* , respectively. The velocity of C, $\underset{\sim}{v}$, and the angular velocity, $\underset{\sim}{\omega}$, of the disk have then the forms:

$$\underset{\sim}{v} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j} + \dot{z}\underset{\sim}{k}$$

$$\underset{\sim}{\omega} = \omega_1\underset{\sim}{n}_1 + \omega_2\underset{\sim}{n}_2 + \omega_3\underset{\sim}{n}_3$$

where $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$ are unit vectors along axes X, Y and Z, respectively. The two triplets of vectors are related by

$$\underset{\sim}{n}_1 = \cos\phi\underset{\sim}{i} + \sin\phi\underset{\sim}{j}$$

$$\underset{\sim}{n}_2 = -\sin\theta\sin\phi\underset{\sim}{i} + \sin\theta\cos\phi\underset{\sim}{j} + \cos\theta\underset{\sim}{k}$$

$$\underset{\sim}{n}_3 = \cos\theta\sin\phi\underset{\sim}{i} - \cos\theta\cos\phi\underset{\sim}{j} + \sin\theta\underset{\sim}{k}$$

The rolling constraint can then be expressed as

$$\underset{\sim}{v} = \underset{\sim}{\omega}\times\underset{\sim}{c}$$

with $\underset{\sim}{c}$ defined as the vector joing P with C, directed from the former to the latter. The foregoing constraint thus yields

$$r c\theta s\phi\dot{\theta} + r\, s\theta c\phi\dot{\phi} + r c\phi\dot{\psi} + \dot{x} = 0$$

$$-r c\theta c\phi\dot{\theta} + r\, s\theta s\phi\dot{\phi} + r s\phi\dot{\psi} + \dot{y} = 0$$

$$r s\theta\dot{\theta} + \dot{z} = 0$$

Integrability of each of these scalar constraints is next verified. These can be rewritten as

$$\underset{\sim}{a}_i^T \dot{\underset{\sim}{q}} = 0, \quad i = 1,2,3$$

with

$$\underset{\sim}{q} = \begin{pmatrix} \theta \\ \phi \\ \psi \\ x \\ y \\ z \end{pmatrix}, \quad \underset{\sim}{a}_1 = r\begin{pmatrix} c\theta s\phi \\ s\theta c\phi \\ c\phi \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \underset{\sim}{a}_2 = r\begin{pmatrix} -c\theta c\phi \\ s\theta s\phi \\ s\phi \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \underset{\sim}{a}_3 = r\begin{pmatrix} s\theta \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus,

$$\frac{\partial \underset{\sim}{a}_1}{\partial \underset{\sim}{q}} = r\begin{bmatrix} -s\theta s\phi & c\theta c\phi & 0 & 0 & 0 & 0 \\ c\theta c\phi & -s\theta s\phi & 0 & 0 & 0 & 0 \\ 0 & -s\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial \underset{\sim}{a}_2}{\partial \underset{\sim}{q}} = r\begin{bmatrix} s\theta c\phi & c\theta s\phi & 0 & 0 & 0 & 0 \\ c\theta s\phi & s\theta c\phi & 0 & 0 & 0 & 0 \\ 0 & c\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial \underset{\sim}{a}_3}{\partial \underset{\sim}{q}} = r \begin{bmatrix} c\theta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Out of these three matrices, only the third one is symmetric. Moreover, $b_i = 0$, for $i = 1,2,3$. Since, additionally, $\underset{\sim}{a}_3$ does not contain time explicitly, the third constraint is integrable, i.e. holonomic, whereas the first two ones are nonholonomic. The integral of the third constraint yields

$$z = r \cos\theta$$

which can be readily verified from Fig 2.2.2.

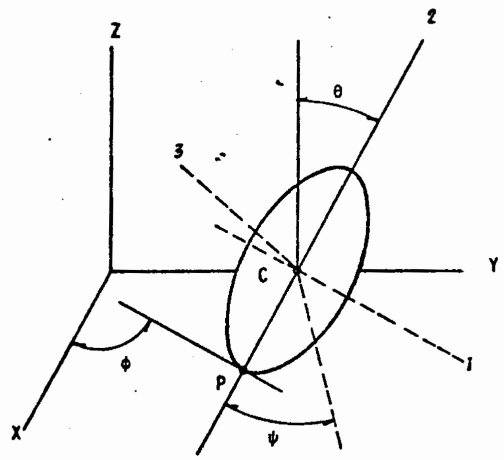

Fig 2.2.2 Disk rolling on a plane.

2.3. Lagrange's form of d'Alembert's Principle for holonomic systems.

As discussed in section 2.1, the only forms of internal energy that a mechanical system can possess are kinetic, potential and nonrecoverable. Potential and nonrecoverable forms of energy of a mechanical system can be regarded as being supplied (or extracted) by the environment, their time rates of change thus being contained in the term U defined in eq. (1.5.11). The first law of Thermodynamics thus takes on the form

$$U = \dot{T} \tag{2.3.1}$$

i.e. the power supplied by the environment to a mechanical system equals the time rate of change of its kinetic energy.

The time rate of change of the kinetic energy of a mechanical system can be obtained by differentiation of eq. (1.4.3). Thus

$$\dot{T} = \sum_{i=1}^{p+r} m_i \dot{\underline{v}}_i \cdot \underline{v}_i + \sum_{j=1}^{r} \left(\dot{\underline{\omega}}_j \cdot \underline{I}_{Cj} \cdot \underline{\omega}_j + \underline{\omega}_j \cdot (\underline{\omega}_j \times \underline{I}_{Cj} \cdot \underline{\omega}_j)\right) \tag{2.3.2}$$

Introducing now eqs. (1.5.9) and (1.5.10a) into eq. (2.3.2), one obtains

$$\dot{T} = \sum_{i=1}^{p+r} m_i \dot{\underline{v}}_i^T \left(\frac{\partial \underline{v}_i}{\partial \dot{\underline{q}}} \dot{\underline{q}} + \underline{\mu}_i\right) + \sum_{j=1}^{r} \left(\underline{I}_{Cj} \dot{\underline{\omega}}_j + \underline{\omega}_j \times \underline{I}_{Cj} \underline{\omega}_j\right)^T \left(\frac{\partial \underline{\omega}_j}{\partial \dot{\underline{q}}} \dot{\underline{q}} + \underline{\nu}_j\right) \tag{2.3.3}$$

This expression can be partitioned then into

$$\dot{T} = \left(\sum_{i=1}^{p+r} m_i \dot{\underline{v}}_i^T \frac{\partial \underline{v}_i}{\partial \dot{\underline{q}}} + \sum_{j=1}^{r} \left(\underline{I}_{Cj} \dot{\underline{\omega}}_j + \underline{\omega}_j \times \underline{I}_{Cj} \underline{\omega}_j\right)^T \frac{\partial \underline{\omega}_j}{\partial \dot{\underline{q}}}\right) \dot{\underline{q}} +$$

$$+ \left(\sum_{i=1}^{p+r} m_i \dot{\underline{v}}_i^T \underline{\mu}_i + \sum_{j=1}^{r} \left(\underline{I}_{Cj} \dot{\underline{\omega}}_j + \underline{\omega}_j \times \underline{I}_{Cj} \underline{\omega}_j\right)^T \underline{\nu}_j\right) \tag{2.3.4}$$

The first term in brackets is readily recognized as $-\phi^{*T}\dot{q}$ by recalling eq. (1.5.16). After introducing eqs. (1.5.14) and (1.5.15) into the second term of the right-hand side of eq. (2.3.4), this is readily recognized to be simply ψ, as defined in eq. (1.5.11a).

Thus,

$$\dot{T} = -\phi^{*T}\dot{q} + \psi \qquad (2.3.5)$$

Substitution of expressions (1.5.11a) and (2.3.5) into eq. (2.3.1) yields, after rearranging of terms,

$$(\phi + \phi^{*})^{T}\dot{q} = 0 \qquad (2.3.6)$$

Now, from the discussion of Sect. 2.2, all functions $\dot{q}_i$ (i=1,...,n) of eq. (2.3.6) are linearly independent. Hence, for eq. (2.3.6) to hold, it is necessary and sufficient that the following holds:

$$\phi + \phi^{*} = 0 \qquad (2.3.7)$$

which is Lagrange's form of d'Alembert's Principle. This expression states that the motion of a holonomic mechanical system of degree of freedom n takes place in such a way that the sum of the n-dimensional vectors of generalized active force and inertia force vanishes.

The vector of generalized inertia force of this system is now manipulatedin order to render it into a more convenient form. This vector is defined in eq. (1.5.15), where each summation contains expressions of the forms $(\partial v/\partial\dot{q})^{T}\dot{v}$ and $(\partial\omega/\partial\dot{q})^{T}(I_C\cdot\dot{\omega}+\omega\times I_C\cdot\omega)$, the indices having been dropped out for compactness. Each of these expressions is next manipulated.

$$\left(\frac{\partial v}{\partial\dot{q}}\right)^{T}\dot{v} = \frac{d}{dt}\left\{\left(\frac{\partial v}{\partial\dot{q}}\right)^{T} v\right\} - \left(\frac{d}{dt}\frac{\partial v}{\partial\dot{q}}\right)^{T} v \qquad (2.3.8)$$

But

$$\left(\frac{\partial v}{\partial \dot{q}}\right)^T v = \frac{\partial}{\partial \dot{q}} \frac{1}{2} v^T v \qquad (2.3.9)$$

and

$$\left(\frac{d}{dt} \frac{\partial v}{\partial \dot{q}}\right)^T v = \left(\frac{d}{dt} \frac{\partial r}{\partial q}\right)^T v = \left(\frac{\partial v}{\partial q}\right)^T v = \frac{\partial}{\partial q} \frac{1}{2} v^T v \qquad (2.3.10)$$

where relation (1.5.2) has been applied and the order of differentiation has been exchanged, which is possible to do under the conditions of the Theorem of Schwarz [6]. Substitution of eqs. (2.3.9) and (2.3.10) into eq. (2.3.8) yields, then

$$\left(\frac{\partial v}{\partial \dot{q}}\right)^T \dot{v} = \frac{d}{dt} \frac{\partial}{\partial \dot{q}} \frac{1}{2} v^T v - \frac{\partial}{\partial q} \frac{1}{2} v^T v \qquad (2.3.11a)$$

or, in operator form,

$$\left(\frac{\partial v}{\partial \dot{q}}\right)^T \dot{v} = \left(\frac{d}{dt} \frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q}\right) \frac{1}{2} v^T v \qquad (2.3.11b)$$

On the other hand,

$$\left(\frac{\partial \omega}{\partial \dot{q}}\right)^T \left(I_C \cdot \dot{\omega} + \omega \times I_C \cdot \omega\right) = \left(\frac{\partial \omega}{\partial \dot{q}}\right)^T \dot{h}_C$$

and

$$\left(\frac{\partial \omega}{\partial \dot{q}}\right)^T \dot{h}_C = \frac{d}{dt}\left[\left(\frac{\partial \omega}{\partial \dot{q}}\right)^T h_C\right] - \left(\frac{d}{dt} \frac{\partial \omega}{\partial \dot{q}}\right)^T h_C \qquad (2.3.12)$$

But

$$\left(\frac{\partial \underset{\sim}{\omega}}{\partial \dot{\underset{\sim}{q}}}\right)^T \underset{\sim}{h}_C = \left(\frac{\partial \underset{\sim}{\omega}}{\partial \dot{\underset{\sim}{q}}}\right)^T \underset{\sim}{I}_C \cdot \underset{\sim}{\omega} = \frac{\partial}{\partial \dot{\underset{\sim}{q}}} \frac{1}{2} \underset{\sim}{\omega}^T \underset{\sim}{I}_C \underset{\sim}{\omega} \tag{2.3.13}$$

and

$$\left(\frac{d}{dt} \frac{\partial \underset{\sim}{\omega}}{\partial \dot{\underset{\sim}{q}}}\right)^T \underset{\sim}{h}_C = \left(\frac{\partial \dot{\underset{\sim}{\omega}}}{\partial \dot{\underset{\sim}{q}}}\right)^T \underset{\sim}{I}_C \cdot \underset{\sim}{\omega} = \left(\frac{\partial \underset{\sim}{\omega}}{\partial \underset{\sim}{q}}\right)^T \underset{\sim}{I}_C \cdot \underset{\sim}{\omega} = \frac{\partial}{\partial \underset{\sim}{q}} \frac{1}{2} \underset{\sim}{\omega}^T \underset{\sim}{I}_C \underset{\sim}{\omega} \tag{2.3.14}$$

where relationship (1.5.10c) was applied. Substitution of eqs. (2.3.13) and (2.3.14) into eq. (2.3.12) yields then

$$\left(\frac{\partial \underset{\sim}{\omega}}{\partial \dot{\underset{\sim}{q}}}\right)^T \left(\underset{\sim}{I}_C \cdot \dot{\underset{\sim}{\omega}} + \underset{\sim}{\omega} \times \underset{\sim}{I}_C \cdot \underset{\sim}{\omega}\right) = \frac{d}{dt} \frac{\partial}{\partial \dot{\underset{\sim}{q}}} \frac{1}{2} \underset{\sim}{\omega}^T \underset{\sim}{I}_C \underset{\sim}{\omega} - \frac{\partial}{\partial \underset{\sim}{q}} \frac{1}{2} \underset{\sim}{\omega}^T \underset{\sim}{I}_C \underset{\sim}{\omega} \tag{2.3.15a}$$

or, in operational form,

$$\left(\frac{\partial \underset{\sim}{\omega}}{\partial \dot{\underset{\sim}{q}}}\right)^T \left(\underset{\sim}{I}_C \cdot \dot{\underset{\sim}{\omega}} + \underset{\sim}{\omega} \times \underset{\sim}{I}_C \cdot \underset{\sim}{\omega}\right) = \left(\frac{d}{dt} \frac{\partial}{\partial \dot{\underset{\sim}{q}}} - \frac{\partial}{\partial \underset{\sim}{q}}\right) \frac{1}{2} \underset{\sim}{\omega}^T \underset{\sim}{I}_C \underset{\sim}{\omega} \tag{2.3.15b}$$

Introducing expressions (2.3.11b) and (2.3.16b) into expression (1.5.16), one obtains

$$\underset{\sim}{\phi}^* = -\sum_{i=1}^{p} \left(\frac{d}{dt} \frac{\partial}{\partial \dot{\underset{\sim}{q}}} - \frac{\partial}{\partial \underset{\sim}{q}}\right) \frac{1}{2} m_i \underset{\sim}{v}_i^T \underset{\sim}{v}_i - \sum_{j=1}^{r} \left(\frac{d}{dt} \frac{\partial}{\partial \dot{\underset{\sim}{q}}} - \frac{\partial}{\partial \underset{\sim}{q}}\right) \frac{1}{2} m_j \underset{\sim}{v}_j^T \underset{\sim}{v}_j$$

$$- \sum_{j=1}^{r} \left(\frac{d}{dt} \frac{\partial}{\partial \dot{\underset{\sim}{q}}} - \frac{\partial}{\partial \underset{\sim}{q}}\right) \frac{1}{2} \underset{\sim}{\omega}_j^T \underset{\sim}{I}_{Cj} \underset{\sim}{\omega}_j \tag{2.3.16}$$

which can in turn be rewritten as

$$\underset{\sim}{\phi}^* = -\left(\frac{d}{dt} \frac{\partial}{\partial \dot{\underset{\sim}{q}}} - \frac{\partial}{\partial \underset{\sim}{q}}\right) \left[\sum_{i=1}^{p} \frac{1}{2} m_i \underset{\sim}{v}_i^T \underset{\sim}{v}_i + \sum_{j=1}^{r} \frac{1}{2} m_j \underset{\sim}{v}_j^T \underset{\sim}{v}_j + \sum_{j=1}^{r} \frac{1}{2} \underset{\sim}{\omega}_j^T \underset{\sim}{I}_{Cj} \underset{\sim}{\omega}_j\right] \tag{2.3.17}$$

The term in brackets is readily recognized as the kinetic energy of the system, T, as defined in eq. (1.4.3). Thus, eq. (2.3.17) becomes

$$\phi^* = -\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} + \frac{\partial T}{\partial q} \qquad (2.3.18)$$

Substitution of eq. (2.3.18) into eq. (2.3.7) yields

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = \phi \qquad (2.3.19)$$

which is a system of n ordinary differential equations governing the motion of a holonomic n-degree of freedom mechanical system. If vector ϕ possesses a potential, it takes on a special form, eqs (2.3.19) thus tranforming accordingly. This form is derived in the next Section.

Example 2.3.1. Dynamical analysis of a differential gear train Reference is made to Fig 2.2.1, where disks A and B are supposed to have masses m_A and m_B, respectively. Moreover, the masses of the shafts are considered to be negligible.

Solution:

The kinetic energy of the system is the sum of the kinetic energy of the three disks. Each of these is next computed.

The kinetic energy of the upper disk A is:

$$T_{AU} = \frac{1}{2}\,\frac{1}{2} m_A a^2 \dot{\theta}_1^2 = \frac{1}{4} m_A a^2 \dot{\theta}_1^2$$

whereas that of the lower one is:

$$T_{AL} = \frac{1}{4} m_A a^2 \dot{\theta}_2^2$$

The kinetic energy of disk B is, in turn:

$$T_B = \frac{1}{2}\omega_B^T I_0 \omega_B$$

with

$$\omega_B = \dot{\theta}_3 n_1 + \dot{\theta}_4 n_2$$

and I_0 is the moment of inertia of the disk with respect to point 0. This is next shown referred to vectors n_1 , n_2 and n_3 , that were defined in Example 2.2.1.

Thus,

$$I_0 = m_B \begin{bmatrix} \frac{1}{2}b^2 & 0 & 0 \\ 0 & a^2+\frac{1}{2}b^2 & 0 \\ 0 & 0 & a^2+\frac{1}{2}b^2 \end{bmatrix}$$

Hence,

$$T_B = \frac{1}{2} m_B \left[\frac{1}{2}b^2 \dot{\theta}_3^2 + (a^2+\frac{1}{2}b^2) \dot{\theta}_4^2 \right]$$

The kinetic energy of the overall system is, then,

$$T = \frac{1}{2}\{\frac{1}{2} m_A a^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) + m_B [\frac{1}{2}b^2 \dot{\theta}_3^2 + (a^2 + \frac{1}{2} b^2) \dot{\theta}_4^2]\}$$

Next, solving for $\dot{\theta}_3$ and $\dot{\theta}_4$ from the constraint equations obtained in Example 2.2.1, one has

$$\dot{\theta}_3 = \frac{a}{2b} (\dot{\theta}_2 - \dot{\theta}_1) , \quad \dot{\theta}_4 = \frac{1}{2} (\dot{\theta}_1 + \dot{\theta}_2)$$

Substitution of the latter values into T yields

$$T = \frac{1}{8} \{m_A a^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) + m_B[\frac{1}{2} a^2(\dot{\theta}_1 - \dot{\theta}_2)^2 + (a^2 + \frac{1}{2} b^2)(\dot{\theta}_1 + \dot{\theta}_2)^2]\}$$

The vector of generalized coordinates has then been reduced to

$$q = [\theta_1 , \theta_2]^T$$

The left-hand side of eq (2.3.19) is now computed.

$$\frac{\partial T}{\partial \dot{\underset{\sim}{q}}} = \frac{1}{4}\begin{Bmatrix} m_A\, a^2\, \dot\theta_1 + m_B[\frac{1}{2}\, a^2(\dot\theta_1 - \dot\theta_2) + (a^2 + \frac{1}{2}\, b^2)(\dot\theta_1 + \dot\theta_2)] \\ m_A\, a^2\, \dot\theta_2 + m_B[-\frac{1}{2}\, a^2(\dot\theta_1 - \dot\theta_2) + (a^2 + \frac{1}{2}\, b^2)(\dot\theta_1 + \dot\theta_2)] \end{Bmatrix}$$

Hence,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\underset{\sim}{q}}} = \frac{1}{4}\begin{Bmatrix} m_A\, a^2\, \ddot\theta_1 + \frac{1}{2}\, m_B\, [(3a^2 + b^2)\ddot\theta_1 + (a^2 + b^2)\ddot\theta_2\,] \\ m_A\, a^2\, \ddot\theta_2 + \frac{1}{2}\, m_B\, [(a^2 + b^2)\ddot\theta_1 + (3a^2 + b^2)\ddot\theta_2] \end{Bmatrix}$$

whereas

$$\frac{\partial T}{\partial \underset{\sim}{q}} = \underset{\sim}{0}$$

Now the right-hand side of eq (2.3.19) is computed. Assuming torques $\underset{\sim}{\ell}_1$ and $\underset{\sim}{\ell}_2$ are applied to the upper and the lower disk A, respectively and that torque $\underset{\sim}{\ell}_3$ acts upon disk B, all three torques directed along axis AA', one has

$$\underset{\sim}{\ell}_1 = \tau_1 \underset{\sim}{n}_2\,, \quad \underset{\sim}{\ell}_2 = \tau_2\, \underset{\sim}{n}_2\,, \quad \underset{\sim}{\ell}_3 = \tau_3\, \underset{\sim}{n}_2$$

Furthermore, let $\underset{\sim}{\omega}_1$ and $\underset{\sim}{\omega}_2$ be the angular velocities of the upper and the lower disk A, and $\underset{\sim}{\omega}_3$ be the angular velocity of disk B. Thus,

$$\underset{\sim}{\omega}_1 = \dot\theta_1\, \underset{\sim}{n}_2, \quad \underset{\sim}{\omega}_2 = \dot\theta_2\, \underset{\sim}{n}_2, \quad \underset{\sim}{\omega}_3 = \dot\theta_3\, \underset{\sim}{n}_1 + \dot\theta_4\, \underset{\sim}{n}_2$$

or, expressing $\underset{\sim}{\omega}_3$ in terms of $\dot\theta_1$ and $\dot\theta_2$,

$$\underset{\sim}{\omega}_3 = \frac{a}{25}\,(\dot\theta_2 - \dot\theta_1)\, \underset{\sim}{n}_1 + \frac{1}{2}\,(\dot\theta_1 + \dot\theta_2)\, \underset{\sim}{n}_2$$

Hence,

$$\frac{\partial \underset{\sim}{\omega}_1}{\partial \underset{\sim}{\dot{q}}} = [\ \underset{\sim}{n}_2\ ,\ \underset{\sim}{0}],\quad \frac{\partial \underset{\sim}{\omega}_2}{\partial \underset{\sim}{\dot{q}}} = [\ \underset{\sim}{0},\ \underset{\sim}{n}_2]\ ,\quad \frac{\partial \underset{\sim}{\omega}_3}{\partial \underset{\sim}{\dot{q}}} = \frac{1}{2}\ [-\frac{a}{b}\ \underset{\sim}{n}_1 + \underset{\sim}{n}_2\ ,\frac{a}{b}\ \underset{\sim}{n}_1 + \underset{\sim}{n}_2\ ]$$

From eq. (1.5.12), then,

$$\underset{\sim}{\phi} = \sum_1^3 (\ \frac{\partial \underset{\sim}{\omega}_i}{\partial \underset{\sim}{\dot{q}}}\ )^T \underset{\sim}{\ell}_i = \begin{pmatrix} \underset{\sim}{n}_2^T \\ \underset{\sim}{0} \end{pmatrix} \tau_1\ \underset{\sim}{n}_2 + \begin{pmatrix} \underset{\sim}{0} \\ \underset{\sim}{n}_2^T \end{pmatrix} \tau_2 \underset{\sim}{n}_2 + \frac{1}{2} \begin{pmatrix} -\frac{a}{b}\ \underset{\sim}{n}_1^T + \underset{\sim}{n}_2^T \\ \frac{a}{b}\ \underset{\sim}{n}_1^T + \underset{\sim}{n}_2^T \end{pmatrix} \tau_3 \underset{\sim}{n}_2$$

$$= \begin{pmatrix} \tau_1 + \frac{1}{2}\tau_3 \\ \tau_2 + \frac{1}{2}\tau_3 \end{pmatrix}$$

The dynamical equations of the system are, then

$$m_A\ a^2\ddot{\theta}_1 + \frac{1}{2}\ m_B[(3a^2 + b^2)\ddot{\theta}_1 + (a^2 + b^2)\ddot{\theta}_2] = 2(2\tau_1 + \tau_3)$$

$$m_A\ a^2\ddot{\theta}_2 + \frac{1}{2}\ m_B[(a^2 + b^2)\ddot{\theta}_1 + (3a^2 + b^2)\ddot{\theta}_2] = 2(2\tau_2 + \tau_3)$$

2.4 Derivation of Lagrange's equation for lamellar holonomic systems.

If the system under study is holonomic and its degree of freedom is n, then its vector of generalized coordinates $\underset{\sim}{q}$ is composed of n independent functions of time. Being lamellar, its vector of generalized force, $\underset{\sim}{\phi}$, arises from a potential $V = V(\underset{\sim}{q}, t)$. Thus

$$\underset{\sim}{\phi} = -\frac{\partial V}{\partial \underset{\sim}{q}} \tag{2.4.1}$$

Substitution of eqs. (2.4.1) and (2.3.18) into eq. (2.3.7) leads to

$$-\frac{\partial V}{\partial \underset{\sim}{q}} - \frac{d}{dt}\frac{\partial T}{\partial \dot{\underset{\sim}{q}}} + \frac{\partial T}{\partial \underset{\sim}{q}} = \underset{\sim}{0} \tag{2.4.2}$$

Now, since $V = V(\underset{\sim}{q}, t)$, one has

$$\frac{\partial V}{\partial \dot{\underset{\sim}{q}}} = \underset{\sim}{0} \tag{2.4.3}$$

and hence

$$\frac{d}{dt}\frac{\partial V}{\partial \dot{\underset{\sim}{q}}} = \underset{\sim}{0} \tag{2.4.4}$$

Adding eq. (2.4.4) to eq. (2.4.2) does not alter it. Hence, rearranging terms,

$$-\frac{d}{dt}\frac{\partial}{\partial \dot{\underset{\sim}{q}}}(T-V) + \frac{\partial}{\partial \underset{\sim}{q}}(T-V) = \underset{\sim}{0} \tag{2.4.5}$$

Defining the Lagrangian of the system as

$$L \equiv T - V \tag{2.4.6}$$

and inverting the signs of both sides of eq. (2.4.5) yields

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{0} \tag{2.4.7}$$

which is a set of *n* differential equations constituting Lagrange's equations of motion for holonomic lamellar mechanical systems. If the potencial V of a lamellar mechanical system does not contain time explicitly, the system is said to be *conservative*. Its dynamical equations are identical to eqs (2.4.7).

Example 2.4.1. Dynamical equations of a conservative mechanical system. A mechanical system composed of a shaft rotating on ball bearings about axis LL' and carrying a slender uniform bar pivoted at O by means of a linear spring of rigidity k (N-m/rad), is shown in Fig 2.4.1. Neglecting the inertia of the shaft, obtain the equations of motion of the system, given that the length and the mass of the bar have values a and m, respectively.

Solution: Define an orthonormal triplet of vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ as shown in Fig 2.4.1. The angular velocity and the inertia dyadic of the bar about point O can be expressed, respectively, as

$$\boldsymbol{\omega} = \dot{\theta}(-c\phi\mathbf{n}_1 + s\phi\mathbf{n}_2) + \dot{\phi}\mathbf{n}_3$$

$$\mathbf{I}_0 = \frac{1}{3} ma^2(\mathbf{n}_2\mathbf{n}_2 + \mathbf{n}_3\mathbf{n}_3)$$

The kinetic energy of the system is, then

$$T = \frac{1}{2}\boldsymbol{\omega}^T\mathbf{I}_0\boldsymbol{\omega} = \frac{1}{6} m a^2(\dot{\theta}^2 s^2\phi + \dot{\phi}^2)$$

The only forces acting on the system are the one due to gravity and that exerted by the spring. The system is thus conservative, its potential being

$$V = -mg\frac{a}{2}c\phi + \frac{1}{2}k\phi^2$$

Its Lagrangian is, then,

$$L = \frac{1}{2}\, ma \left[\frac{a}{3}\left(\dot{\theta}^2 s^2\phi + \dot{\phi}^2\right) + g c\phi\right] - \frac{1}{2} k \phi^2$$

Define its vector of generalized coordinates as

$$\underset{\sim}{q} = [\theta\ ,\ \phi]^T$$

Hence,

$$\frac{\partial L}{\partial \dot{\underset{\sim}{q}}} = \frac{ma^2}{3} \begin{bmatrix} \dot{\theta}\, s^2\phi \\ \dot{\phi} \end{bmatrix}, \quad \frac{\partial L}{\partial \underset{\sim}{q}} = \begin{bmatrix} 0 \\ \frac{ma}{6}\left(2a\dot{\theta}^2 c\phi - 3g\right)s\phi - k\phi \end{bmatrix}$$

The first of eqs. (2.4.7) then yields

$$\frac{d}{dt}\ \frac{ma^2}{3}\ \dot{\theta}s^2\phi = 0$$

i.e.

$$\dot{\theta}s^2\phi = \alpha = \text{const}$$

The second of these equations yields, in turn,

$$\ddot{\phi} - \frac{1}{2a}\left(2a\dot{\theta}^2 c\phi - 3g\right) s\phi - \frac{3k}{ma^2}\phi = 0$$

or, equivalently,

$$\ddot{\phi} - \left(\alpha c\phi - \frac{3gs\phi}{2a}\right) - \frac{3k}{ma^2}\phi = 0$$

which is a nonlinear second order ordinary differential equation governing the motion of the system. The value of constant α is to be determined from initial conditions. Once ϕ is known, θ is obtained by integration as

$$\theta(t) = \int_0^t \frac{\alpha\, dt}{\sin^2\phi(t)}$$

That the second-degree-of-freedom system produced only one

1st order ODE is due to the fact that L does not contain θ explicitly. θ is then referred to as an *ignorable* coordinate, or a *cyclic* coordinate.

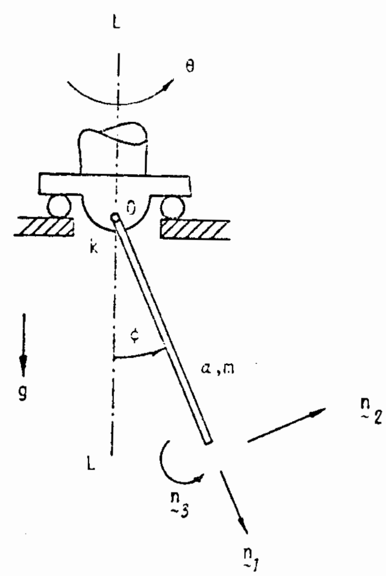

Fig 2.4.1 Bar suspended from shaft rotating on ball-bearings

2.5 Extension of Lagrange's equations to nonlamellar systems.

If some forces and couples acting on a given holonomic system arise from a potential, whereas some others do not, the vector of generalized force, ϕ, can be decomposed as shown in eq. (2.1.10), i.e. as

$$\phi = -\frac{\partial V}{\partial q} + \phi_{np} \tag{2.5.1}$$

Substitution of eqs. (2.5.1) and (2.4.12) into eq. (2.3.7) yields then

$$-\frac{\partial V}{\partial q} + \phi_{np} - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}} + \frac{\partial T}{\partial q} = 0 \tag{2.5.2}$$

Now, considering eq. (2.4.15) and definition (2.4.17), eq. (2.5.2) transforms into

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \phi_{np} \tag{2.5.3}$$

which is the set of Lagrange's equations applied to a holonomic nonlamellar system whose degree of freedom is n, this being the dimension of its vector of generalized coordinates.

Example 2.5.1. Dynamical analysis of a nonconservative system. Given the system shown in Fig 2.4.1, assume that a torque τ is applied to its shaft, but otherwise it remains unchanged. Derive its new equations of motion.

Solution: Eqs. (2.5.3) are now to be applied. The Lagrangian of the new system is identical to that established for Example 2.4.1. Hence, the left-hand side of eqs (2.5.3) is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \frac{ma^2}{3}\begin{bmatrix} \ddot{\theta}s^2\phi + 2\dot{\theta}\dot{\phi}\, s\phi c\phi \\ \\ \ddot{\phi} \end{bmatrix} - \begin{bmatrix} \\ \\ \frac{ma}{6}(2a\dot{\theta}^2 c\phi - 3g)s\phi - k\phi \end{bmatrix}$$

Its right-hand side is now computed. The applied torque can be expressed in vector form as

$$\ell = -\tau c\phi\, n_1 + \tau s\phi\, n_2$$

The angular velocity of the bar was obtained in Example 2.4.1 as

$$\omega = \dot{\theta}(-c\phi\, n_1 + s\phi\, n_2) + \dot{\phi}\, n_3$$

Hence

$$\frac{\partial \omega}{\partial \dot{q}} = [\,-c\phi n_1 + s\phi n_2\,,\ n_3]$$

and so,

$$\phi_{np} = \begin{bmatrix} -c\phi n_1^T + s\phi n_2^T \\ \\ n_3^T \end{bmatrix} (-\tau c\phi n_1 + \tau s\phi n_2) = \begin{bmatrix} \tau \\ \\ 0 \end{bmatrix}$$

The dynamical equations sought are, then

$$\ddot{\theta}s^2\phi + 2\dot{\theta}\dot{\phi}\ s\phi c\phi = 3\tau/ma^2$$

$$\ddot{\phi} - (\dot{\theta}^2 c\phi - \frac{3g}{2a})s\phi + \frac{3k}{ma^2}\phi = 0$$

Notice that, if a potentiometer is introduced into the system to measure ϕ , as well as tachometers to measure $\dot{\theta}$ and $\dot{\phi}$, these equations can be used to estimate the value of τ, which can be needed in turn to close the loop of a control system used to regulate, say, the velocity of the shaft.

2.6. Equations of motion for nonholonomic system

If a mechanical system possessing a n-dimensional vector of generalized coordinates, $\underset{\sim}{q}$, is subject to the m nonholonomic constraints

$$\underset{\sim}{A}\,\dot{\underset{\sim}{q}} + \underset{\sim}{b} = \underset{\sim}{0} \qquad (2.6.1)$$

where A is a $m \times n$ matrix function of $\underset{\sim}{q}$ and t, then Lagrange's equations (2.4.7) are no longer applicable. To cope with this situation several procedures have been devised. The classical one consists of defining a functional whose minimization leads to the desired equations of motion, incorporating the nonholonomic constraints (2.6.1) via the introduction of Lagrange multipliers [5,6]. This approach, however, is lengthy, for it demands the computation of constraint forces represented by the Lagrange multipliers. Two more interesting approaches have been made to avoid such computations [7,8], out of which derives the next formulation.

To begin with, partition matrix $\underset{\sim}{A}$ and vector $\underset{\sim}{q}$ of eq (2.6.1) into the form

$$\underset{\sim}{A} = \overset{\quad p \qquad\quad m}{\left(\underset{\sim}{A}_{mp} \;\vdots\; \underset{\sim}{A}_{m} \right)} \; m \quad , \quad \underset{\sim}{q} = \begin{pmatrix} \underset{\sim}{q}_p \\ \hline \underset{\sim}{q}_m \end{pmatrix} \begin{matrix} p \\ m \end{matrix}$$

where $p \equiv n - m$.

Eq (2.6.1) can thus be rewritten as

$$\underset{\sim}{A}_{mp}\dot{\underset{\sim}{q}}_p + \underset{\sim}{A}_m\dot{\underset{\sim}{q}}_m + \underset{\sim}{b} = \underset{\sim}{0} \qquad (2.6.2)$$

In many instaces, $\underset{\sim}{q}$ contains Euler's angles, but the computation of the kinetic energy, T, requires not the derivatives of these, but the components of the angular velocity of each rigid body, which are linear combinations of those derivatives. Such components of angular velocity can be grouped within vector $\underset{\sim}{u}$, defined next as a linear combination of the components of $\dot{\underset{\sim}{q}}$, i.e.

$$\underset{\sim}{u} \equiv \underset{\sim}{C}_p \dot{\underset{\sim}{q}}_p + \underset{\sim}{C}_{pm} \dot{\underset{\sim}{q}}_m \tag{2.6.3}$$

Now, defining vector $\underset{\sim}{w}$ and matrix $\underset{\sim}{C}$ as

$$\underset{\sim}{w} = \begin{pmatrix} \underset{\sim}{u} \\ \underset{\sim}{b} \end{pmatrix} \begin{matrix} p \\ m \end{matrix} \quad , \quad \underset{\sim}{C} = \begin{pmatrix} \underset{\sim}{C}_p & \underset{\sim}{C}_{pm} \\ \underset{\sim}{A}_{mp} & \underset{\sim}{A}_m \end{pmatrix} \begin{matrix} p \\ m \end{matrix}$$

$$\qquad\qquad\qquad\qquad p \qquad m$$

eqs. (2.6.2) and (2.6.3) can be written in compact form as

$$\underset{\sim}{w} = \underset{\sim}{C} \dot{\underset{\sim}{q}} \tag{2.6.5}$$

from which, if both $\underset{\sim}{C}_p$ and $\underset{\sim}{A}_m$ are invertible, $\dot{\underset{\sim}{q}}$ can be readily obtained as

$$\dot{\underset{\sim}{q}} = \underset{\sim}{C}^{-1} \underset{\sim}{w} \tag{2.6.6}$$

with

$$\underset{\sim}{C}^{-1} = \begin{pmatrix} \underset{\sim}{X} & \underset{\sim}{Y} \\ \underset{\sim}{Z} & \underset{\sim}{W} \end{pmatrix} \begin{matrix} p \\ m \end{matrix} \tag{2.6.7}$$

$$\qquad p \qquad m$$

where

$$\underset{\sim}{X} \equiv (\underset{\sim}{C}_p - \underset{\sim}{C}_{pm} \underset{\sim}{A}_m^{-1} \underset{\sim}{A}_{mp})^{-1} \quad , \quad \underset{\sim}{Y} \equiv -\underset{\sim}{X}\underset{\sim}{C}_{pm}\underset{\sim}{A}_m^{-1}$$

$$\underset{\sim}{W} \equiv (\underset{\sim}{A}_m - \underset{\sim}{A}_{mp} \underset{\sim}{C}_p^{-1} \underset{\sim}{C}_{pm})^{-1} \quad , \quad \underset{\sim}{Z} \equiv -\underset{\sim}{W}\underset{\sim}{A}_{mp}\underset{\sim}{C}_p^{-1} \tag{2.6.8}$$

Thus,

$$\dot{\underset{\sim}{q}}_p = \underset{\sim}{X}\underset{\sim}{u} - \underset{\sim}{Y}\underset{\sim}{b} \; , \; \dot{\underset{\sim}{q}}_m = \underset{\sim}{Z}\underset{\sim}{u} - \underset{\sim}{W}\underset{\sim}{b} \tag{2.6.9}$$

Hence,

$$\frac{\partial \dot{q}_p}{\partial u} = X \ , \ \frac{\partial \dot{q}_m}{\partial u} = Z \tag{2.6.10a}$$

$$\frac{\partial \dot{q}_p}{\partial b} = - Y \ , \ \frac{\partial \dot{q}_m}{\partial b} = - W \tag{2.6.10b}$$

Now, if f is any N-dimensional vector function of q and $\dot{q}$, one has

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial \dot{q}_p} \frac{\partial \dot{q}_p}{\partial u} + \frac{\partial f}{\partial \dot{q}_m} \frac{\partial \dot{q}_m}{\partial u} = \frac{\partial f}{\partial \dot{q}_p} X + \frac{\partial f}{\partial \dot{q}_m} Z \tag{2.6.11a}$$

and

$$\frac{\partial f}{\partial b} = \frac{\partial f}{\partial \dot{q}_p} \frac{\partial \dot{q}_p}{\partial b} + \frac{\partial f}{\partial \dot{q}_m} \frac{\partial \dot{q}_m}{\partial b} = - \frac{\partial f}{\partial \dot{q}_p} Y - \frac{\partial f}{\partial \dot{q}_m} W \tag{2.6.11b}$$

The reduced equations of motion for nonholonomic systems are now derived. To this end, use is made of the First Law of Thermodynamics, eq. (2.3.1), where U, the power supplied to the system by its environment, is computed as

$$U = \phi^T \dot{q} + \psi = \dot{q}^T \phi + \psi \tag{2.6.12a}$$

whereas $\dot{T}$, as

$$\dot{T} = \phi^{*T} \dot{q} + \psi = \dot{q}^T \phi^* + \psi \tag{2.6.12b}$$

The first term of U is thus a sum of terms either of the form $\dot{q}^T(\partial v_i/\partial \dot{q})^T f_i$ or of the form $\dot{q}^T(\partial \omega_i/\partial \dot{q})^T \ell_{Ci}$. The first term of T is, correspondingly, a sum of terms of either of the foregoing forms, except that f_i and ℓ_{Ci} are substituted by f_i^* and ℓ_{Ci}^*, as given by eqs. (1.5.14) and (1.5.15), respectively. In what follows, only the first type of terms is expanded. Thus,

$$\dot{q}^T \left(\frac{\partial v_i}{\partial \dot{q}}\right)^T f_i = [\dot{q}_p^T \; \vdots \; \dot{q}_m^T] \begin{bmatrix} \left(\frac{\partial v_i}{\partial \dot{q}_p}\right)^T \\ \hline \left(\frac{\partial v_i}{\partial \dot{q}_m}\right)^T \end{bmatrix} f_i =$$

$$= \left(\frac{\partial v_i}{\partial \dot{q}_p} \dot{q}_p + \frac{\partial v_i}{\partial \dot{q}_m} \dot{q}_m\right)^T f_i \qquad (2.6.13)$$

Introducing eq. (2.6.9) into eq. (2.6.13), one obtains

$$\dot{q}^T \left(\frac{\partial v_i}{\partial \dot{q}}\right)^T f_i = \left[\frac{\partial v_i}{\partial \dot{q}_p} (Xu - Yb) + \frac{\partial v_i}{\partial \dot{q}_m} (Zu - Wb) \right]^T f_i =$$

$$= u^T \left(\frac{\partial v_i}{\partial \dot{q}_p} X + \frac{\partial v_i}{\partial \dot{q}_m} Z \right)^T f_i + b^T \left(- \frac{\partial v_i}{\partial \dot{q}_p} Y - \frac{\partial v}{\partial \dot{q}} W \right)^T f_i$$

Next, eqs. (2.6.11a & b) are introduced into the latter equation, with f substituted by v_i. Thus,

$$\dot{q}^T \left(\frac{\partial v_i}{\partial \dot{q}}\right)^T f = u^T \left(\frac{\partial v_i}{\partial u}\right)^T f_i + b^T \left(\frac{\partial v_i}{\partial b}\right)^T f_i \qquad (2.6.14a)$$

Similarly, one has for the second type of terms,

$$\dot{q}^T \left(\frac{\partial \omega_i}{\partial \dot{q}}\right)^T \ell_{Ci} = u^T \left(\frac{\partial \omega_i}{\partial u}\right)^T \ell_{Ci} + b^T \left(\frac{\partial \omega_i}{\partial b}\right)^T \ell_{Ci} \qquad (2.6.14b)$$

with analogous expressions for inertia forces and moments. Substitution of expressions (2.6.14a & b) into eq. (2.6. 12a) yields

$$U = \underset{\sim}{u}^T\left[\sum_{1}^{p+r}\left(\frac{\partial \underset{\sim}{v}_i}{\partial \underset{\sim}{u}}\right)^T \underset{\sim}{f}_i + \sum_{1}^{r}\left(\frac{\partial \underset{\sim}{\omega}_i}{\partial \underset{\sim}{u}}\right)^T \underset{\sim}{\ell}_{Ci}\right] + \underset{\sim}{b}^T\left[\sum_{1}^{p+r}\left(\frac{\partial \underset{\sim}{v}_i}{\partial \underset{\sim}{b}}\right)^T \underset{\sim}{f}_i + \sum_{1}^{r}\left(\frac{\partial \underset{\sim}{\omega}_i}{\partial \underset{\sim}{b}}\right)^T \underset{\sim}{\ell}_{Ci}\right] + \psi \tag{2.6.15}$$

The first term in brackets is analogous to $\underset{\sim}{\phi}$, as defined in eq. (1.5.12), except that instead of being of dimension n, it is of dimension $p<n$; hence, it seems justified to call it the *reduced vector of generalized force* and is henceforth represented by $\underset{\sim}{f}$. The second term in brackets has no analogous term by far. It will be represented by $\underset{\sim}{g}$, its dimension being m, also less than n. Eq. (2.6.15) thus can be expressed as

$$U = \underset{\sim}{u}^T\underset{\sim}{f} + \underset{\sim}{b}^T\underset{\sim}{g} + \psi \tag{2.6.16}$$

Paralelling the foregoing discussion, the following expression is readily obtained for $\dot{T}$:

$$\dot{T} = -\underset{\sim}{u}^T\underset{\sim}{f}^* - \underset{\sim}{b}^T\underset{\sim}{g}^* + \psi \tag{2.6.17}$$

where $\underset{\sim}{f}^*$ and $\underset{\sim}{g}^*$ are defined similarly to vectors $\underset{\sim}{f}$ and $\underset{\sim}{g}$, except that instead of referring to active forces and moments, $\underset{\sim}{f}_i$ and $\underset{\sim}{\ell}_{Ci}$, respectively, they refer to inertia forces and moments $\underset{\sim}{f}_i^*$ and $\underset{\sim}{\ell}_{Ci}^*$. $\underset{\sim}{f}^*$ is correspondingly referred to as the reduced inertia generalized force, its dimension being $p<n$.

Substitution of eqs. (2.6.16) and (2.6.17) into the First Law of Thermodynamics, eq. (2.3.1), yields

$$\underset{\sim}{u}^T\underset{\sim}{f} + \underset{\sim}{b}^T\underset{\sim}{g} = -\underset{\sim}{u}^T\underset{\sim}{f}^* - \underset{\sim}{b}^T\underset{\sim}{g}^* \tag{2.6.18}$$

from which

$$\underset{\sim}{u}^T(\underset{\sim}{\delta} + \underset{\sim}{\delta}^*) + \underset{\sim}{b}^T(\underset{\sim}{g} + \underset{\sim}{g}^*) = 0 \qquad (2.6.19)$$

readily follows. Now, the second term in paranthesis in eq. (2.6.19) is expanded

$$\underset{\sim}{g} + \underset{\sim}{g}^* = \sum_{1}^{p+r} \left(\frac{\partial \underset{\sim}{v}_i}{\partial \underset{\sim}{b}}\right)^T \underset{\sim}{f}_i + \sum_{1}^{r}\left(\frac{\partial \underset{\sim}{\omega}_i}{\partial \underset{\sim}{b}}\right)^T \underset{\sim}{\ell}_{Ci} +$$

$$+\left[\sum_{1}^{p+r}\left(\frac{\partial \underset{\sim}{v}_i}{\partial \underset{\sim}{b}}\right)^T \underset{\sim}{f}_i^* + \sum_{1}^{r}\left(\frac{\partial \underset{\sim}{\omega}_i}{\partial \underset{\sim}{b}}\right)^T \underset{\sim}{\ell}_{Ci}^*\right] =$$

$$= \sum_{1}^{p+r}\left(\frac{\partial \underset{\sim}{v}_i}{\partial \underset{\sim}{b}}\right)^T (\underset{\sim}{f}_i + \underset{\sim}{f}_i^*) + \sum_{1}^{r}\left(\frac{\partial \underset{\sim}{\omega}_i}{\partial \underset{\sim}{b}}\right)^T (\underset{\sim}{\ell}_{Ci} + \underset{\sim}{\ell}_{Ci}^*) \qquad (2.6.20)$$

From Newton's Second Law, eqs (1.6.3 a- c), each term $\underset{\sim}{f}_i + \underset{\sim}{f}_i^*$, as well as $\underset{\sim}{\ell}_{Ci} + \underset{\sim}{\ell}_{Ci}^*$, in eq (2.6.20), vanishes. Hence, eq. (2.6.19) becomes

$$\underset{\sim}{u}^T(\underset{\sim}{\delta} + \underset{\sim}{\delta}^*) = \underset{\sim}{0} \qquad (2.6.21)$$

However, all components of vector $\underset{\sim}{u}$ have been assumed to be linearly independent functions of time. Hence, for eq. (2.6.21) to hold, for arbitrary values of $\underset{\sim}{u}$, the sum in paranthesis must vanish, i.e.

$$\underset{\sim}{\delta} + \underset{\sim}{\delta}^* = \underset{\sim}{0} \qquad (2.6.22)$$

which is the Lagrangian form of d'Alembert's Principle for a p-degree-of-freedom nonholonomic mechanical system subject to the m nonholonomic constraints given by eq. (2.6.1).

Next, a relationship between vectors $\underset{\sim}{\delta}$ and $\underset{\sim}{\phi}$ or, correspondingly, between $\underset{\sim}{\delta}^*$ and $\underset{\sim}{\phi}^*$, is derived. $\underset{\sim}{\delta}$ being a sum of terms of the forms

$(\partial v_i/\partial u)^T f_i$ and $(\partial \omega_i/\partial u)^T \ell_{Ci}$, only the first ones will be expanded, i.e.

$$\left(\frac{\partial v_i}{\partial u}\right)^T f_i = \left(\frac{\partial v_i}{\partial \dot{q}_p} X + \frac{\partial v_i}{\partial \dot{q}_m} Z\right)^T f_i \qquad (2.6.23)$$

where use has been made of eq. (2.6.11a) with v_i instead of f. The term in brackets, however, can be expressed as

$$\frac{\partial v_i}{\partial \dot{q}_p} X + \frac{\partial v_i}{\partial \dot{q}_m} Z = \left[\frac{\partial v_i}{\partial \dot{q}_p} \;\vdots\; \frac{\partial v_i}{\partial \dot{q}_m}\right]\begin{bmatrix} X \\ -- \\ Z \end{bmatrix} = \frac{\partial v_i}{\partial \dot{q}}\begin{bmatrix} X \\ -- \\ Z \end{bmatrix} \qquad (2.6.24)$$

Substitution of eq. (2.6.24) into eq. (2.6.23) now yields

$$\left(\frac{\partial v_i}{\partial u}\right)^T f_i = \left[X^T \;\vdots\; Z^T\right]\left(\frac{\partial v_i}{\partial \dot{q}}\right)^T f_i \qquad (2.6.25)$$

At the light of relationship (2.6.25), it should be clear now that

$$\delta \equiv \sum_{1}^{p+r} \left(\frac{\partial v_i}{\partial u}\right)^T f_i + \sum_{1}^{r} \left(\frac{\partial \omega_i}{\partial u}\right)^T \ell_{Ci} =$$

$$= \left[X^T \;\vdots\; Z^T\right]\left[\sum_{1}^{p+r}\left(\frac{\partial v_i}{\partial \dot{q}}\right)^T f_i + \sum_{1}^{r}\left(\frac{\partial \omega_i}{\partial \dot{q}}\right)^T \ell_{Ci}\right] \qquad (2.6.26)$$

The term in brackets is readily identified as ϕ. Hence the following

$$\delta = \left[X^T \;\vdots\; Z^T\right] \phi \qquad (2.6.27a)$$

and, similarly,

$$\delta^* = \left[X^T \;\vdots\; Z^T\right] \phi^* \qquad (2.6.27b)$$

Now, from eq. (2.4.12), which is an identity that holds regardless of the nature of the system -i.e. eq. (2.4.12) holds for mechanical systems whether holonomic or nonholonomic -, one obtains

$$\mathbf{f}^* = -[\mathbf{X}^T \mid \mathbf{Z}^T]\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{q}}} - \frac{\partial T}{\partial \mathbf{q}}\right) \tag{2.6.28}$$

Substitution of eq (2.6.28) into eq (2.6.22) leads to

$$[\mathbf{X}^T \mid \mathbf{Z}^T]\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{q}}} - \frac{\partial T}{\partial \mathbf{q}}\right) = \mathbf{f} \tag{2.6.29}$$

which is the counterpart of eq (2.3.19). Now, combining eqs (2.6.27a) and (2.6.29), one obtains

$$[\mathbf{X}^T \mid \mathbf{Z}^T][\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{q}}} - \frac{\partial T}{\partial \mathbf{q}} - \boldsymbol{\phi}] = \mathbf{0} \tag{2.6.29a}$$

Eqs (2.6.29a) constitute a system of p independent ordinary differential equations taking into account the m nonholonomic constraints (2.6.2). Integration of the former with given corresponding initial values of the p independent coordinates, and then solution of the latter for the remaining m dependent coordinates produces the solution to the dynamical problem proposed.

Vectors $\mathbf{f}$ and $\mathbf{f}^*$ defined in (2.6.27a & b) are p ($<n$)- dimensional. Hence it seems natural to refer to them as the *reduced* vectors of active and inertial generalized forces, respectively.

Now, if $\boldsymbol{\phi}$ arises from lamellar fields, i.e. if a potential $V=V(\mathbf{q},t)$ exists such that

$$\boldsymbol{\phi} = -\frac{\partial V}{\partial \mathbf{q}}$$

eq (2.6.27a) yields

$$\mathbf{f} = -[\mathbf{X}^T \mid \mathbf{Z}^T]\frac{\partial V}{\partial \mathbf{q}} \tag{2.6.30}$$

Substitution of eq (2.6.30) into eq (2.6.29) leads to

$$[\underset{\sim}{X}^T \vdots \underset{\sim}{Z}^T]\left\{ \frac{d}{dt}\frac{\partial T}{\partial \dot{\underset{\sim}{q}}} - \frac{\partial T}{\partial \underset{\sim}{q}} + \frac{\partial V}{\partial \underset{\sim}{q}} \right\} = 0 \tag{2.6.31}$$

By resorting to similar arguments as for deriving eq (2.4.7), one obtains from eq (2.6.31)

$$[\underset{\sim}{X}^T \vdots \underset{\sim}{Z}^T]\left\{ \frac{d}{dt}\frac{\partial L}{\partial \dot{\underset{\sim}{q}}} - \frac{\partial L}{\partial \underset{\sim}{q}} \right\} = \underset{\sim}{0} \tag{2.6.32}$$

which is a system of p Lagrange's equations governing the motion of a p-degree of freedom lamellar nonholonomic mechanical system.

Matrices $\underset{\sim}{X}$ and $\underset{\sim}{Z}$ appearing in eq (2.6.32) can simplify under special circumstances, which brings about an illuminating form of the said equation. In fact, if vector $\underset{\sim}{u}$, defined in eq (2.6.3) is made identical to vector $\underset{\sim}{q}_p$, then $\underset{\sim}{C}_p$ becomes $\underset{\sim}{1}_p$, the $p \times p$ identity matrix, $\underset{\sim}{C}_{pm}$ becoming the $p \times m$ zero matrix. In this case, then

$$\underset{\sim}{X} = \underset{\sim}{1}_p \quad \text{and} \quad \underset{\sim}{Z} = -\underset{\sim}{A}_m \underset{\sim}{A}_{mp}.$$

Now, if the second factor of the right-hand side of eq (2.6.32) is partitioned accordingly, this equation can be rewritten as

$$[\underset{\sim}{1}_p \vdots -\underset{\sim}{A}_{mp}^T \ (\underset{\sim}{A}_m^T)^{-1}] \begin{Bmatrix} \frac{d}{dt}\frac{\partial L}{\partial \dot{\underset{\sim}{q}}_p} - \frac{\partial L}{\partial \underset{\sim}{q}_p} \\ \hdashline \frac{d}{dt}\frac{\partial L}{\partial \dot{\underset{\sim}{q}}_m} - \frac{\partial L}{\partial \underset{\sim}{q}_m} \end{Bmatrix} = \underset{\sim}{0} \tag{2.6.32a}$$

Assuming that L be smooth enough as to fulfill the hypotheses of Schwarz's Theorem [4], the next two identities follow:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_p} = \frac{\partial \dot{L}}{\partial \dot{q}_p}, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_m} = \frac{\partial \dot{L}}{\partial \dot{q}_m} \tag{2.6.33a}$$

where

$$\dot{L} \equiv \frac{dL}{dt} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial t} \tag{2.6.33b}$$

At this point a caveat is in order: Misled by the results shown in eqs (1.5.2) and (1.5.10c), one might think that eq (2.6.33b) implies that $\partial\dot{L}/\partial\dot{q}$ is identically equal to $\partial L/\partial q$. This is not so, however, for, in differentiating $\dot{L}$, as given by eq (2.6.33b), both partial derivatives $\partial L/\partial\dot{q}$ and $\partial L/\partial t$ contain $\dot{q}$ explicitly. In fact, equating $\partial\dot{L}/\partial\dot{q}$ with $\partial L/\partial q$ yields Lagrange's equation for holonomic lamellar systems.

Eq (2.6.32) is now expanded and identities (2.6.33a) are introduced in the resulting expansion. One obtains:

$$\frac{\partial \dot{L}}{\partial \dot{q}_p} - A_{mp}^T (A_m^T)^{-1} \frac{\partial \dot{L}}{\partial \dot{q}_m} - \left[\frac{\partial L}{\partial q_p} - A_{mp}^T (A_m^T)^{-1} \frac{\partial L}{\partial q_m} \right] = 0 \tag{2.6.34}$$

On the other hand, since q_m is a (possibly implicit) function of q_p and time, one can write.

$$q_m = q_m (q_p, t) \tag{2.6.35}$$

Thus, L can be regarded as function only of q_p and t, for which reason one can define a "total" derivative of L with respect to q_p as

$$\frac{dL}{dq_p} \equiv \frac{\partial L}{\partial q_p} + \left(\frac{\partial q_m}{\partial q_p} \right)^T \frac{\partial L}{\partial q_m} \tag{2.6.36a}$$

and, similarly,

$$\frac{d\hat{L}}{d\dot{\underset{\sim}{q}}_p} \equiv \frac{\partial\hat{L}}{\partial\dot{\underset{\sim}{q}}_p} + \left(\frac{\partial\dot{\underset{\sim}{q}}_m}{\partial\dot{\underset{\sim}{q}}_p}\right)^T \frac{\partial\hat{L}}{\partial\dot{\underset{\sim}{q}}_m} \tag{2.6.36b}$$

Now, eq (2.6.35) yields

$$\dot{\underset{\sim}{q}}_m = \frac{\partial\underset{\sim}{q}_m}{\partial\underset{\sim}{q}_p}\dot{\underset{\sim}{q}}_p + \frac{\partial\underset{\sim}{q}_m}{\partial t} \tag{2.6.37}$$

from which, since $\dot{\underset{\sim}{q}}_m$ is independent from $\underset{\sim}{q}_m$, one obtains

$$\frac{\partial\dot{\underset{\sim}{q}}_m}{\partial\dot{\underset{\sim}{q}}_p} = \frac{\partial\underset{\sim}{q}_m}{\partial\underset{\sim}{q}_p} \tag{2.6.38}$$

On the other hand, eq (2.6.2) yields

$$\dot{\underset{\sim}{q}}_m = -\underset{\sim}{A}_m^{-1}(\underset{\sim}{A}_{mp}\dot{\underset{\sim}{q}}_p + \underset{\sim}{b}) \tag{2.6.39}$$

from which

$$\frac{\partial\dot{\underset{\sim}{q}}_m}{\partial\dot{\underset{\sim}{q}}_p} = -\underset{\sim}{A}_m^{-1}\underset{\sim}{A}_{mp} \tag{2.6.39a}$$

Substitution of eqs (2.6.26a & b, 38 and 39a) into eq (2.6.34) yields now

$$\frac{d\hat{L}}{d\dot{\underset{\sim}{q}}_p} - \frac{dL}{d\underset{\sim}{q}_p} = \underset{\sim}{0}\text{ , or }\frac{d}{dt}\frac{dL}{d\dot{\underset{\sim}{q}}_p} - \frac{dL}{d\underset{\sim}{q}_p} = \underset{\sim}{0} \tag{2.6.40}$$

which is an illuminating form of Lagrange's equations, similar to eqs (2.4.7), except that eqs (2.6.40) are valid for lamellar nonholonomic systems, taking into account the m nonholonomic constraints (2.6.1). Eqs (2.6.40) thus refer only to the independent coordinates of the nonholonomic system.

If the nonholonomic system at hand is acted upon some (generalized) forces arising from a potential, and some other ones not possessing any potential, then a reasoning similar to that leading to eqs (2.5.3) yields

$$[\underset{\sim}{X}^T \vdots \underset{\sim}{Z}^T][\frac{d}{dt}\frac{\partial L}{\partial \dot{\underset{\sim}{q}}} - \frac{\partial L}{\partial \underset{\sim}{q}} - \underset{\sim}{\phi}_{np}] = 0 \qquad (2.6.41)$$

where $\underset{\sim}{\phi}_{np}$ comprises the vector of generalized force not arising from any potential. The foregoing results are natural extensions to results previously reported in [8].

Example 2.6.1. Analysis of the rolling disk. This is a classical problem that is very often resorted to in order to illustrate methods of analysis of nonholonomic dynamical systems. Its analysis using different techniques appears in [5-8]. The method presented here is now applied to solve this problem. Reference is now made to the system shown in Fig 2.2.2, whose constraint equations were obtained in Example 2.2.2. If z is substituted by $r \cos\theta$, the vector of generalized variables of the system reduces to

$$\underset{\sim}{q} = [\ \theta,\ \phi,\ \psi,\ x,\ y]^T$$

Since the system now possesses no holonomic constraints, but two nonholonomic ones, its degree of freedom is three. Hence, any three of the five components of vector $\underset{\sim}{q}$ can be regarded as independent. For their nature, however, it seems plausible to consider the angular variables as independent. Thus, vector $\underset{\sim}{q}$ is partitioned accordingly as

$$\underset{\sim}{q}_{\delta} = \begin{bmatrix} \theta \\ \phi \\ \psi \end{bmatrix}, \quad \underset{\sim}{q}_{m} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Now, returning to the constraint equations derived in Example 2.2.2, expressed in the form of eq (2.6.2), one has, with $p=3$ and $m=2$,

$$\mathbf{A}_{23}\,\mathbf{q}_3 + \mathbf{A}_2\mathbf{q}_2 = \mathbf{0}$$

where

$$\mathbf{A}_{23} = r\begin{bmatrix} c\theta s\phi & s\theta s\phi & c\phi \\ -c\theta c\phi & s\theta s\phi & s\phi \end{bmatrix}$$

and $\mathbf{A}_2 = \mathbf{1}_2$, i.e. the 2×2 identity matrix. Moreover, the only external force acting upon the system is gravity. Hence, the system is conservative, its equations of motion then taking on the form of eqs (2.6.34) or, equivalently, of eqs (2.6.40).

To derive the aforementioned equations of motion, its kinetic and potential energies are needed. The kinetic energy is

$$T = \frac{1}{2}\,m\mathbf{v}^2 + \frac{1}{2}\,\boldsymbol{\omega}^T\mathbf{I}_C\boldsymbol{\omega}$$

$\mathbf{v}$ and $\boldsymbol{\omega}$ having been defined in Example 2.2.2. Other variables are m and $\mathbf{I}_C$, the mass of the disk and its inertia tensor with respect to the mass center C, respectively, i.e.

$$\mathbf{I}_C = \frac{1}{4}\,mr^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

which is referred to axes 1,2 and 3 shown in Fig 2.2.2. Performing computations, one has

$$T = \frac{1}{2} m \{\dot{x}^2 + \dot{y}^2 + \frac{r^2}{4} [(1+4s^2\theta)\dot{\theta}^2 + (c^2\theta + 2s^2\theta)\dot{\phi}^2 + 4\dot{\psi}\dot{\phi}s\theta + 2\dot{\psi}^2]\}$$

The potential energy is

$$V = mgrc\theta$$

Hence,

$$\frac{\partial L}{\partial \dot{q}_3} = mr^2 \begin{bmatrix} (1+4s^2\theta)\dot{\theta}/4 \\ (c^2\theta + 2s^2\theta)\dot{\phi}/4 + s\theta\dot{\psi}/2 \\ s\theta\dot{\phi}/2 + \dot{\psi}/2 \end{bmatrix}, \quad \frac{\partial L}{\partial \dot{q}_2} = m \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

$$\frac{\partial L}{\partial q_3} = \frac{1}{2} mr^2 \begin{bmatrix} 2s\theta c\theta\dot{\theta}^2 + s\theta c\theta\dot{\phi}^2/2 + c\theta\dot{\psi}\dot{\phi} + 2gs\theta/r \\ 0 \\ 0 \end{bmatrix}, \quad \frac{\partial L}{\partial q_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Substitution of the foregoing expressions into eq (2.6.34) yields, after dropping the connon factor mr,

$$r \begin{bmatrix} (1+4s^2\theta)\ddot{\theta}/4 + 2s\theta c\theta\dot{\theta}^2 \\ (1+s^2\theta)\ddot{\phi}/4 + s\theta c\theta\dot{\theta}\dot{\phi}/2 + s\theta\ddot{\psi}/2 + c\theta\dot{\psi}\dot{\theta}/2 \\ s\theta\ddot{\phi}/2 + c\theta\dot{\phi}\dot{\theta}/2 + \ddot{\psi}/2 \end{bmatrix} +$$

$$+r \begin{bmatrix} s\theta c\theta\dot{\theta}^2 + s\theta c\theta\dot{\phi}^2/4 + c\theta\dot{\psi}\dot{\phi}/2 + gs\theta/r \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} c\theta s\phi & -c\theta c\phi \\ s\theta c\phi & s\theta s\phi \\ c\psi & s\psi \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

where, from the nonholonomic constraints,

$$\ddot{x}/r = -(c\theta s\phi\ddot{\theta} + s\theta c\phi\ddot{\phi} + c\phi\ddot{\psi}) - (-s\theta s\phi\dot{\theta}^2 + 2c\theta c\phi\dot{\theta}\dot{\phi} - s\theta s\phi\dot{\phi}^2 - s\phi\dot{\psi}\dot{\phi})$$

$\ddot{y}/r = -(-c\theta c\phi\ddot{\theta} + s\theta s\phi\ddot{\phi} + s\phi\ddot{\psi}) - (s\theta c\phi\dot{\theta}^2 + 2c\theta s\phi\dot{\theta}\dot{\phi} + s\theta c\phi\dot{\phi}^2 + c\phi\dot{\psi}\dot{\phi})$

The equations of motion thus reduce to

$$5\ddot{\theta} - 5\, s\theta c\theta\dot{\phi}^2 - 6\, c\theta\dot{\psi}\dot{\phi} - 4\, g\, s\theta/r = 0 \qquad \text{(a)}$$

$$(1+5s^2\theta)\ddot{\phi} + 10 s\theta c\theta\dot{\theta}\dot{\phi} + 6 s\theta\ddot{\psi} + 2\, c\theta\dot{\psi}\dot{\theta} = 0 \qquad \text{(b)}$$

$$3\, s\theta\ddot{\phi} + 3\ddot{\psi} + 5\, c\theta\, \dot{\theta}\dot{\phi} = 0 \qquad \text{(c)}$$

Elimination of $\ddot{\psi}$ from the last two equation just derived leads to

$$c\theta\ddot{\phi} + 2\dot{\psi}\dot{\theta} = 0 \qquad \text{(b')}$$

Equations (a), (b') and (c) constitute the equations sought, which coincide with the results published in the literature.

Example 2.6.2. Dynamical analysis of a rolling two wheel axle. A two-wheel axle is shown in Fig. 2.6.1. It consists of two identical rigid disks, each of mass m and radius r, mounted on a rigid shaft AB of length ℓ and negligible mass. If the planes of the disks are perpendicular to line AB and each disk can rotate freely about the shaft, determine the motion of the system considering given initial conditions and that the axle rolls without slipping on a horizontal surface.

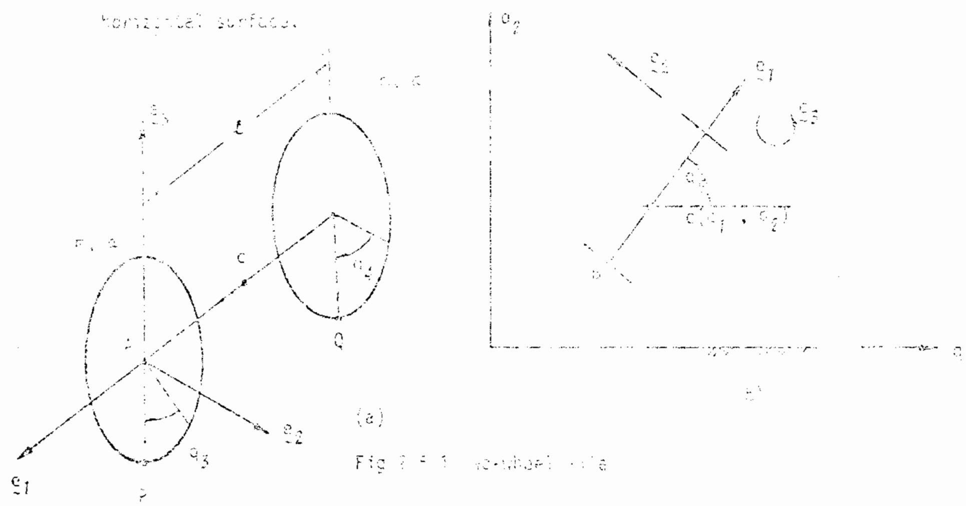

Fig 2.6.1 Two-wheel axle

Solution:

Let the vector of generalized coordinates be

$$\underset{\sim}{q} = [q_1 , q_2 , q_3 , q_4 , q_5]^T$$

each q_i $(i=1,\ldots,5)$ being shown in Fig 2.6.1, where C, with Cartesian coordinates q_1 and q_2 , is the middle point of line AB. Furthermore, let $\underset{\sim}{v}_A$ and $\underset{\sim}{v}_B$ be the velocities of points A and B, respectively, $\underset{\sim}{\omega}_3$ and $\underset{\sim}{\omega}_4$ being the angular velocities of the disks associated with q_3 and q_4 , respectively. The constraints of the system are obtained next:

From the fact that line AB is rigid, one has

$$\underset{\sim}{v}_C = \frac{1}{2} (\underset{\sim}{v}_A + \underset{\sim}{v}_B)$$

which yields, with $c_5 = \cos q_5$ and $s_5 = \sin q_5$,

$$\dot{q}_1 = -\frac{1}{2} a(\dot{q}_3 + \dot{q}_4) c_5 , \dot{q}_2 = \frac{1}{2} a(\dot{q}_3 + \dot{q}_4) s_5$$

On the other hand, the relative velocity of point B with respect to A can be obtained in two different ways: either regarding these points as belonging to each of the disks or regarding them as points of bar AB. This yields

$$\dot{q}_5 = \frac{a}{\ell} (\dot{q}_4 - \dot{q}_3)$$

The foregoing three equations on the components of $\dot{\underset{\sim}{q}}$ constitute the kinematic constraints of the system. Since the third one involves constant coefficients only, it is integrable, its integration producing

$$q_5 = \frac{a}{\ell} (q_4 - q_3)$$

under the assumption that at some given configuration all three angles vanish. The first two constraint equations are next analysed for integrability. These can be rewritten in the form:

$$2\dot{q}_1 + ac_5\,\dot{q}_3 + ac_5\,\dot{q}_4 = 0$$

$$2\dot{q}_2 - as_5\,\dot{q}_3 - as_5\,\dot{q}_4 = 0$$

which can be written in turn as

$$\underset{\sim}{a}_1^T\,\dot{\underset{\sim}{q}} = 0\ ,\quad \underset{\sim}{a}_2^T\,\dot{\underset{\sim}{q}} = 0$$

with

$$\underset{\sim}{a}_1 = [2,\ 0,\ ac_5\ ,\ ac_5\ ,\ 0]^T,\quad \underset{\sim}{a}_2 = [\ 0,\ 2,\ -as_5\ ,\ -as_5\ ,\ 0]^T$$

The partial derivatives of these vectors are:

$$\frac{\partial \underset{\sim}{a}_1}{\partial \underset{\sim}{q}} = [\underset{\sim}{0}_{54}\,\vdots\,\underset{\sim}{v}_1\ ]\ ,\quad \frac{\partial \underset{\sim}{a}_2}{\partial \underset{\sim}{q}} = [\ \underset{\sim}{0}_{54}\,\vdots\,\underset{\sim}{v}_2\ ]$$

where $\underset{\sim}{0}_{54}$ is the $5x4$ zero matrix, vectors $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ being

$$\underset{\sim}{v}_1 = [0,\ 0,\ -\ as_5\ ,\ -\ as_5\ ,\ 0]^T\ ,\quad \underset{\sim}{v}_2 = [\ 0,\ 0,\ -\ ac_5\ ,\ -\ ac_5\ ,\ 0\ ]^T$$

thereby showing that neither matrix is symmetric. These constraints are, then, nonholonomic, the system thus having a double degree of freedom. At this point one should notice that the motion of the system could be reduced to that in the 1-2 plane, Fig 2.6.1 (b). This figure makes apparent that the motion of the system reduces to that of line AB in the plane. A line moving on a plane, however, possesses a triple degree of freedom, as is well known from elementary mechanics. The fact that the system proposed possesses only a double degree of freedom can be realized by noticing that line AB of Fig 2.6.1 (b) is prevented from sliding along itself.

In order to derive the equations of motion, vector $\underset{\sim}{q}$ is now redefined and partitioned as follows

$$\underset{\sim}{q} = [\, q_3 \, , \, q_4 \, , \, q_1 \, , \, q_2 \,]^T \, , \, \underset{\sim}{q}_p = [\, q_3 \, , \, q_4 \,]^T \, , \, \underset{\sim}{q}_m = [\, q_1 \, , \, q_2 \,]^T$$

where q_3 and q_4 have been chosen to be the independent coordinates, with $m = p = 2$. The constraint equations can then be rewritten in form (2.6.2) with

$$\underset{\sim}{A}_{mp} = a \begin{bmatrix} c_5 & c_5 \\ -s_5 & -s_5 \end{bmatrix}, \; \underset{\sim}{A}_m = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2\, \underset{\sim}{1}_2$$

Next, $\underset{\sim}{c}_p$ is chosen to be $\underset{\sim}{1}_2$, the 2x2 unit matrix, whereas $\underset{\sim}{c}_{pm}$ is defined as the 2x2 zero matrix. Hence,

$$\underset{\sim}{X} = \underset{\sim}{1}_2 \, , \, \underset{\sim}{W} = \underset{\sim}{A}_m^{-1} = \frac{1}{2}\, \underset{\sim}{1}_2 \, , \, \underset{\sim}{Z} = -\frac{a}{2} \begin{bmatrix} c_5 & c_5 \\ -s_5 & -s_5 \end{bmatrix}$$

The only force acting upon the system is gravity which, due to the restrictions imposed, does not produce any power. Hence the system is conservative and eqs (2.6.32a) can be applied. One has

$$T = \frac{1}{4} m\ell^2 [(3+\alpha^2)\dot{q}_3^2 - 2\alpha^2 \dot{q}_3 \dot{q}_4 + (3+\alpha^2)\dot{q}_4^2] \, , \, V = 0$$

with $\alpha \equiv a/\ell$

Hence,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\underset{\sim}{q}}_p} = \frac{1}{2} m\ell^2 \begin{bmatrix} (3+\alpha^2)\ddot{q}_3 - \alpha^2 \ddot{q}_4 \\ -\alpha^2 \ddot{q}_3 + (3+\alpha^2)\ddot{q}_4 \end{bmatrix}.$$

$$\frac{\partial L}{\partial \underset{\sim}{q}_p} = \underset{\sim}{0}, \quad \frac{\partial L}{\partial \dot{\underset{\sim}{q}}_m} = \underset{\sim}{0}, \quad \frac{\partial L}{\partial \underset{\sim}{q}_m} = \underset{\sim}{0}$$

Eqs (2.6.32a) thus yield

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_p} - \frac{\partial L}{\partial q_p} = \underset{\sim}{0}$$

i.e., in component form:

$$(3 + \alpha^2)\ddot{q}_3 - \alpha^2\ddot{q}_4 = 0$$

$$-\alpha^2\ddot{q}_3 + (3+\alpha^2)\ddot{q}_4 = 0$$

which produces

$$(3+\alpha^2)\dot{q}_3 - \alpha^2\dot{q}_4 = \omega_1$$

$$-\alpha^2\dot{q}_3 + (3+\alpha^2)\dot{q}_4 = \omega_2$$

where ω_1 and ω_2 are constants. Hence,

$$\dot{q}_3 = \frac{1}{3}[(3+\alpha^2)\omega_1 + \alpha^2\omega_2],\ \dot{q}_4 = \frac{1}{3}[\alpha^2\omega_1 + (3+\alpha^2)\omega_2]$$

and hence

$$\dot{q}_5 = \frac{3+2\alpha^2}{3}(\omega_1 + \omega_2)\alpha$$

i.e., with $q_5(0) \equiv 0$,

$$q_5 = \frac{3+2\alpha^2}{3}(\omega_1 + \omega_2)\alpha t = \Omega t$$

The nonholonomic constraints yield

$$\dot{q}_1 = -\frac{3+2\alpha^2}{6}(\omega_1 + \omega_2)\cos q_5\ ,\ \dot{q}_2 = \frac{3+2\alpha^2}{6}(\omega_1 + \omega_2)\sin q_5$$

Integration of the later equations produces

$$q_1 = -\frac{1}{2\alpha}\sin \Omega t,\ q_2 = -\frac{1}{2\alpha}(\cos\Omega t - 1)$$

with $q_1(0) \equiv 0$, $q_2(0) \equiv 0$. Point C thus moves describing a circle on the *1-2* plane, centered at point $(0,\ 1/2\alpha)$, with radius $1/2\alpha$.

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